RANDOM GRAPHS ON THE HYPERBOLIC PLANE

by

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ABSTRACT

In this thesis, we study a recently proposed model of random graphs that exhibit properties which are present in a wide range of networks arising in real world settings. The model creates random geometric graphs on the hyperbolic plane, where vertices are connected if they are within a certain threshold distance. We study typical properties of these graphs.

We identify two critical values for one of the parameters that act as sharp thresholds. The three resulting intervals of the parameters that correspond to three possible phases of the random structure: A.a.s., the graph is connected; A.a.s., the graph is not connected, yet there is a giant component; A.a.s., every component is of sublinear size. Furthermore, we determine the behaviour at the critical values.

We also consider typical distances between vertices and show that the ultra-small world phenomenon is present. Our results imply that most pairs of vertices that belong to the giant component are within doubly logarithmic distance.

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Chapter 1

Introduction

1.1 Random Graphs

The study of random graphs was introduced in a seminal paper by Erdős and Rényi [ER60], and independently in a paper by Gilbert [Gil59]. Two very similar models were developed, $G(N, M)$ and $G(N, p)$. The model $G(N, M)$ consists of N vertices with $0 \leq M \leq \binom{N}{2}$ edges. A graph is sampled uniformly among all the graphs with precisely M edges. $G(N, p)$ also has N vertices, but here every possible edge is present with probability p, independent of all the other edges. Due to the strong concentration of the binomial distribution, $G(N, M)$ and $G(N, p)$ behave very similarly for $p = \binom{N}{2}/M$ (see the book by Janson, Łuczak and Ruciński [JŁR00]). The first two papers by Erdős and Rényi [ER60, ER59] investigated the typical component structure of $G(N, p)$. They discovered two thresholds for p at which the structure of the graphs changes significantly. In particular, the connectivity of the graph changes around $p = \frac{\log(N)}{N}$. All results in this thesis hold *asymptotically almost surely* (a.a.s.), that is, with a probability that tends to 1 as the number N of vertices grows. For the following we fix any $\varepsilon > 0$. For $p < \frac{(1-\varepsilon)\log N}{N}$, a.a.s. the graph $G(N, p)$ contains isolated vertices, thus it is not connected. On the other hand, for $p > \frac{(1+\varepsilon)\log N}{N}$, a.a.s. the graph is connected. The other threshold discovered by Erdős and Rényi is around $p = \frac{1}{N}$. For $p < \frac{1-\varepsilon}{N}$, a.a.s. the largest component in $G(N, p)$ has order proportional to $\log N$, whereas a.a.s.

there is a unique *giant component*, of order linear in N, for $p > \frac{1+\varepsilon}{N}$. In between these thresholds, when $p = \frac{1}{N}$, the order of the largest component scales as $n^{\frac{2}{3}}$.

A useful concept to further study the structural changes that happen in the graph as more edges are present is the random graph process $\tilde{G}_N$. Here, starting with a graph on N vertices and no edges, the edges are added one by one, chosen uniformly at random among all the non-edges for each step. This process allows for a more refined analysis of the phase transition with regards to the largest component of the graph. Bollobás [Bol84] showed that, when the giant component emerges, a gap arises between the largest and second largest component. In particular, while the order of the largest component cannot decrease as more edges are added, a.a.s. the order of the second largest component decreases. He further shows that when the number of edges is close to the threshold $\frac{N}{2}$, adding a single edge increases the size of the largest component by four vertices on average, capturing the strong threshold that appears around this value. Bollobás and Thomason [BT85] used the graph process to further analyse the threshold for the connectivity of the random graph. It turns out that a.a.s. adding the edge that removes the last isolated vertex is precisely the one that yields connectivity in the graph.

Achlioptas processes give an alternative way of choosing edges that is not uniform. Here, in every step, two potential edges are chosen uniformly at random and exactly one of them is added to the graph, according to some rule. Achlioptas proposed this model with the question whether there are rules that delay the emergence of the giant component. Bohman and Frieze [BF01] showed that this indeed works for a very simple rule. Given the option of edges e and e', the edge e' is only chosen if it is disjoint from all previously chosen edges. This small bias towards connecting previously isolated vertices is enough to postpone the emergence of the giant component by a constant factor of N.

For a thorough introduction to random graphs, see e.g. the books of Bollobás [Bol01] or Janson, Łuczak and Ruciński [JŁR00].

1.2 Random Geometric Graphs

Gilbert [Gil61] originally introduced a model for random geometric graphs on an infinite space in the context of continuum percolation. Here, vertices are generated using a *point process*: the number of vertices in any given measurable area follows a Poisson distribution with a mean proportional to the measure of the area. Any two vertices are then connected by an edge if they are within some fixed distance δ. The density of the point process or the distance that determines whether vertices are adjacent can equivalently be used as the parameter of the model. A central question in this setting is whether the resulting graph has an infinite component, an infinite equivalent to the question whether a finite graph contains a component of linear size.

Finite random geometric graphs were first studied by Hafner [Haf72]. These are formed by randomly distributing N vertices on some finite subset of a metric space and then connecting all those pairs of vertices with an edge that are within distance δ, for $\delta > 0$. In general, $\delta = \delta(N)$ decreases as N increases, as a way to keep the average degree fixed (possibly with respect to N). Usually, the domain in which the graph is created is fixed and $\delta(N)$ is used as a parameter. These graphs are motivated by an abundance of real-world problems that can be modelled by them, e.g. networks of transmitters, each of which can communicate with every other transmitter within its range. The prototypical and probably most studied model is that of graphs on a d-dimensional real-valued unit-torus, using the euclidean norm on the torus to measure distances. Vertices are typically independently uniformly distributed.

While the methods for studying random geometric graphs necessarily differ from the ones used in Erdős-Rényi random graphs, many properties can be examined in the geometric model as well. It turns out that there are critical values $r_c^C(N)$ and $r_c^G(N)$ so that the following hold for a random geometric graph G on N vertices on $[0,1]^2$ with threshold distance r (see Penrose [Pen03]). Fix $\varepsilon > 0$. A.a.s., G is connected if $r > (1+\varepsilon)r_c^C(N)$, whereas it is not connected for $r < (1-\varepsilon)r_c^C(N)$. A.a.s., G has a component of linear size if $r > (1+\varepsilon)r_c^G(N)$, whereas there is no such component for $r < (1-\varepsilon)r_c^G(N)$. Unlike $r_c^C(N)$, the exact value of $r_c^G(N)$ is unknown as it is directly tied to the analogue critical value in continuum percolation that determines whether there is an infinite component, which is currently unknown.

For more information on random geometric graphs, see the book by Penrose [Pen03].

1.3 Complex Networks

While euclidean random geometric graphs work well as a model for many applications, there is a wide range of naturally occurring networks that have common properties, not all of which are satisfied in the euclidean model. Examples of these networks include the Internet, collaboration networks and airport networks. The main properties that characterize these networks are the following:

1. *Sparseness*: The number of edges is proportional to the number of vertices.

2. *Small World Phenomenon*: Even though the graph is sparse, almost all pairs of vertices in the same component have a relatively short (logarithmic or doubly logarithmic) graph distance.

3. *Clustering*: Two vertices with a common neighbour are much more likely to be adjacent than a random pair of vertices.

4. *Scale Free Degree Distribution*: The tail of the distribution follows a power law.

Having a power-law degree distribution means that there is a constant $\gamma > 0$ so that the proportion of vertices with a given degree k scales roughly like $k^{-\gamma}$. Networks that exhibit these properties are usually called *complex networks*. Evidence suggests that the exponent γ usually lies between 0 and 3. For a more complete discussion of these properties see Chung and Lu [CL06].

As the class of complex networks is very broad, it would be very useful to have a model to sample such networks randomly, to study their properties and create efficient algorithms. The models previously covered in this chapter are all *homogeneous*, i.e., all their vertices behave roughly the same, that is without knowledge about other vertices or edges, every vertex has the same expected degree and these degrees are moderately concentrated. Complex networks however exhibit some degree of inhomogeneity. In particular, typical examples display ‘hubs’, a small set of vertices that have a very large degree, in contrast to the average degree being constant.

Euclidean random geometric graphs naturally exhibit clustering. However, they cannot both have the small world phenomenon and sparseness. This is essentially caused by the fact that in euclidean spaces the area of a ball increases polynomially with the radius. Thus a small threshold distance to connect vertices, which is needed to have constant average degrees, implies that the graph distance between most pairs must be very large. The homogeneity of the graphs also prevents a scale free degree distribution. In the last decades, several models have arisen that aim to feature as many of the properties as possible and their study is getting increasing attention.

The *Preferential Attachment Model* was introduced by Barabási and Albert [BA99] as a first model for scale free networks. Vertices are added one by one and each vertex is connected to m previously added vertices by an edge, where m is constant and the parameter of the graph that determines the average degree of the graph, which is roughly $2m$. The neighbours are chosen randomly, each vertex has a probability that is proportional to its current degree. This process naturally boosts the degree of vertices that already have a high degree, thus creating the hubs that are vital to complex networks. Bollobás, Riordan, Spencer and Tusnády [BRST01] proved that the degree distribution follows a power law with exponent $\gamma = 3$. Bollobás and Riordan [BR04] showed that preferential attachment graphs on N vertices typically have an average distance between vertices that scales like $\frac{\log N}{\log \log N}$, giving a small world phenomenon. Bollobás [Bol03] determined the clustering coefficient in preferential attachment graphs. It turns out that this value is a function tending to 0 as N tends to ∞, while evidence suggests that complex networks exhibit a constant clustering coefficient. Various other properties and versions of the preferential attachment graph have been studied and remain a popular research area (see e.g. [Mór05] and [EN11]).

Another model that has been extensively studied and is of particular interest for this thesis was introduced by Chung and Lu [CL03]. Here, every vertex is assigned a weight, and a pair of vertices is connected by an edge with a probability that is proportional to the product of its weights, independently of the other pairs. Chung and Lu prove that, given suitably chosen weights, the degree distribution follows a power law. In fact, the expected degree of a vertex corresponds to its weight, so the weights should be chosen to represent a power law. For exponents in the power law between 2 and 3, average distances are with high probability close to $\log \log N$. However, as the edges are chosen independently, clustering cannot be observed.

In the following section we will introduce the model that we study in this thesis. It was proposed by Krioukov, Papadopoulus, Kitsak, Vahdat and Boguñá [KPK+10]. They assume that hyperbolic geometry underlies complex networks and gives rise to the properties that can be observed within them. These networks exhibit a hierarchical structure that resembles a tree, nodes form groups, that in turn are organised into subgroups, etc. The hyperbolic space is a natural candidate to accommodate such tree-like structures. This can be (informally) observed for example in uniform tilings of the hyperbolic plane, see [Hat02].

The analysis of Krioukov et al. [KPK+10] indicates that the degree distribution of vertices in their model follows a power law and they give empirical data that suggests that clustering is present. Gugelmann, Panagiotou and Peter [GPP12] verified these results rigorously, and Candellero and Fountoulakis [CF] further studied clustering in a variant of the model.

In a way, this model is an equivalent to the Chung-Lu model as a random geometric graph in an inhomogeneous space, inheriting the properties from the Chung-Lu model but acquiring clustering from its nature as a geometric graph. Fountoulakis [Fou15] studied these similarities. Some function of the distance of a vertex from the origin essentially functions as the weight used in the Chung-Lu model. The expected degree of a vertex scales like this value, and, if only the distances to the origin are exposed, the probability of two vertices being connected via an edge is the product of these respective values.

Kiwi and Mitsche [KM15] showed that the expected diameter of the graph is polylogarithmic in N for $\frac{1}{2} < \alpha < 1$, with an exponent depending on α. This and one of our results determining the average distances, given in Chapter 5, confirm that a small world phenomenon is present in this model.

1.4 Random Graphs on the Hyperbolic Plane

The hyperbolic plane $\mathbb{H} = \mathbb{H}_1$ is an unbounded surface of constant curvature -1. There are many ways to represent it as a two dimensional plane with a given metric, including the half-plane model, the Beltrami-Klein disk model and the Poincaré disk model. The Poincaré disk model is the unit disc $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, equipped with the metric determined by the differential form $ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$. In the hyperbolic plane of curvature $-\gamma^2$, the differential becomes $ds^2 = \frac{4(dx^2+dy^2)}{\zeta^2(1-x^2-y^2)^2}$. Here, the circumference of a circle of radius r is $2\pi\frac{1}{\zeta}\sinh(\alpha r)$, whereas the are is $2\pi\frac{1}{\zeta^2}(\cosh(\alpha R) - 1)$.

The *native model* we use in this thesis to draw pictures was introduced by Krioukov et al. and is a variant of the Poincaré disk model that projects the disc onto the whole of $\mathbb{R}^2$. We use polar coordinates, where every point in the Poincaré disk retains their angle, but the radius becomes the hyperbolic distance to the origin. While this representation lacks some of the useful geometric properties that the Poincaré disk and the other models offer, it has the advantage that the change in density within any image is not as large as in the Poincaré disk. In fact, if we were to draw interesting graphs, i.e. sparse graphs as required by the properties of copmplex networks, on the Poincaré disk, all of their vertices would be on to the periphery of the disc, making it almost impossible to see which vertices are closer to the origin, a vital information.

The random graph model we consider in this thesis is the Krioukov-Papadopoulos-

Kitsak-Vahdat-Boguñá model, or KPKVB model. It was introduced by Krioukov et al. and involves two parameters, ν and α. The graph $\mathcal{G}(N;\alpha,\nu)$ is formed by randomly choosing N vertices on the disc $\mathcal{D}_R$ of radius $R := 2\log(N/\nu)$ in $\mathbb{H}$ according to the following *quasi-uniform distribution*:

If the random point u has polar coordinates (r,θ), then θ, r are independent, θ is uniformly distributed in $(0,2\pi]$ and the probability distribution of r has density function given by:

$$\rho(r) = \begin{cases} \alpha\frac{\sinh\alpha r}{\cosh\alpha R-1} & \text{if } 0 \le r \le R, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

This corresponds to the ratio of circumference of a circle of radius r in the hyperbolic plane of curvature $-\alpha^2$ to the area of such a circle. Thus, when $\alpha = 1$, this distribution is uniform on $\mathcal{D}_R$. In any other case, however, we can think of it as the uniform distribution on a disc of radius R on the space $\mathbb{H}_\alpha$ of curvature $-\alpha^2$, then transferring angles and radii to $\mathcal{D}_R$ on $\mathbb{H}$. The distribution is essentially an exponential one, and α acts as the logarithm of the base. The effect of this, as we will see later, is that with a larger α vertices tend to be located towards the periphery of the disc, whereas a smaller α draws them towards the origin.

Two vertices in $\mathcal{G}(N;\alpha,\nu)$ are connected by an edge whenever they are within hyperbolic distance R. The choice of R as the threshold distance seems unintuitive, as this would yield a very dense graph in the euclidean geometry. However, $\mathcal{D}_R$ is very dense at its periphery, so dense that two vertices on the periphery will have to be in a very thin section of $\mathcal{D}_R$ to be connected. In fact, the distance being R allows for the vertices in the centre to have a very high degree, whereas the vertices close to the periphery have constant expected degree. Figure 1.1 shows an example of such a random graph on $N = 1000$ vertices.

We should mention that Krioukov et al. in fact had an additional parameter ζ in their definition of the model. In this model, the points are taken inside a disc of radius

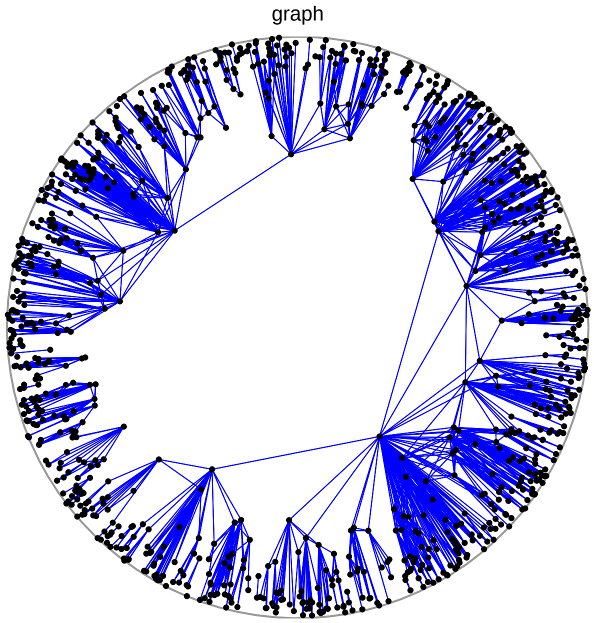

Figure 1.1: Simulation of the KPKVB-model with $N = 1000, \alpha = .9, \nu = 2$. (Depicted in the native model.)

$R_\zeta := (2/\zeta)\log(N/\nu)$ on the hyperbolic plane $\mathbb{H}_\zeta$ of curvature $-\zeta^2$, and the points are generated according to (1.1) with R_ζ in place of R. In this case the random graph is denoted by $\mathcal{G}(N;\zeta,\alpha)$. However, it turns out that there is no need for the extra parameter ζ. The following lemma, which we prove in Appendix A.1, shows that we can take $\zeta = 1$ without any loss of generality. We remind the reader that a *coupling* of two random objects X, Y is a common probability space for a pair of objects (X', Y') whose marginal distributions satisfy $X' \overset{d}{=} X, Y' \overset{d}{=} Y$.

Lemma 1.4.1. *Let $\alpha, \alpha', \zeta, \zeta' > 0$ be such that $\zeta/\alpha = \zeta'/\alpha'$. For every ν and $N \in \mathbb{N}$, there exists a coupling between $\mathcal{G}(N;\zeta,\alpha,\nu)$ and $\mathcal{G}(N;\zeta',\alpha',\nu)$ such that $G(N;\zeta,\alpha,\nu) = G(N;\zeta',\alpha',\nu)$.*

In other words, the previous lemma states that one can define the random graphs $\mathcal{G}(N;\zeta,\alpha,\nu)$ and $\mathcal{G}(N;\zeta',\alpha',\nu)$ on a common probability space in such a way that the two graphs are isomorphic (with probability one). Let us also remark that *the edge-set of $\mathcal{G}(N;\alpha,\nu)$ is decreasing in α and increasing in ν* in the following precise sense.

Lemma 1.4.2. *Let $\alpha, \alpha', \nu, \nu' > 0$ be such that $\alpha \geq \alpha'$ and $\nu \leq \nu'$. For every $N \in \mathbb{N}$, there exists a coupling such that $G(N; \alpha, \nu)$ is a subgraph of $G(N; \alpha', \nu')$.*

The proof of Lemma 1.4.2 is given in Appendix A.2.

1.5 Notation

We use standard Landau Notation. For two non-negative functions $f(x), g(x)$ with $g(x) \neq 0$, we write $f(x) = o(g(x))$ or $g(x) \sim f(x)$ if $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$ and $f(x) = O(g(x))$ if there exists a constant $C > 0$ such that $\frac{f(x)}{g(x)} < C$ $\forall x > 0$. If $f(x) = O(g(x))$ and $g(x) = O(f(x)$ we write $f(x) = \Theta(g(x))$ or $f(x) \approx g(x)$.

When we say an event $\mathcal{E}_N$ holds asymptotically almost surely (abbreviated a.a.s.), we mean that $\mathbb{P}(\mathcal{E}_N) \to 1$ as $N \to \infty$.

One important step that makes $\mathcal{G}(N; \alpha, \nu)$ easier to work with is using angles instead of distances. We denote by $\mathrm{dist}_{\mathbb{H}}(u, v)$ the hyperbolic distance of two points u and v in $\mathbb{H}$. We use $\theta_{u,v}$ for the relative angle between two vertices u and v, i.e. the angle enclosed by the rays Ou and Ov, where O is the origin of $\mathcal{D}_R$. Sometimes we want to only consider vertices that are on one side of the ray Ou. It is then convenient to talk of the clockwise or anticlockwise relative angles $\theta_c(u, v)$ and $\theta_a(u, v)$. The terms clockwise and anticlockwise are generally used when all pairs of vertices in question have very small relative angle, making it obvious which direction a vertex is of another.

We denote by $|L_1|$ the size of a largest connected component of $\mathcal{G}(N; \alpha, \nu)$. For a given vertex u, we denote by $C(u)$ the component containing u. When we talk about graph distance as opposed to the geometric distance, we use the notation $d_G(u, v)$ for the graph distance of u and v. We use $u \sim v$ to denote that the vertices u and v are adjacent. When talking about adjacency, we use the terms vertex and point interchangeably, i.e., two vertices or points are adjacent if they are within distance R.

1.6 Results

In this thesis, we prove results on the component structure and distances in KPKVB graphs. In Chapter 2, we state some preliminary results and tools that we will use throughout the thesis. In particular, working with hyperbolic distances is hard, so we state essential tools to give nice bounds on this distance, given radii of vertices, as well as their relative angle.

In chapters 3 and 4, we study a "phase transition" in the component structure of the graphs, depending on the parameter α. The values $\alpha = 1/2$ and $\alpha = 1$ prove to be critical points in this transition. When $\alpha < 1/2$, the graph is connected a.a.s., whereas there are isolated vertices for $\alpha > 1/2$ (which is a straightforward consequence of the results on the degree sequence by Gugelmann et al. [GPP12]). However, as long as $\alpha < 1$, there is a component of size proportional to N. When $\alpha > 1$, this component disappears and all components are sublinear in N.

This behaviour is made precise in the following three theorems. Note that, as $R = 1/2 \log N$, the following statements can be expressed in terms of just N. However, it is convenient for readability to use both terms R and N.

Theorem 1.6.1. *Let α, ν be positive real numbers. The following hold:*

- *if $\alpha > 1$, then a.a.s. $|L_1| < 8R^2 \log^3(R)\ N^{1/\alpha}$.*
- *if $\alpha < 1$, then there exists $c = c(\alpha, \nu) > 0$ such that a.a.s. $|L_1| > cN$.*

Recently, Kiwi and Mitsche [KM15] showed that the second largest component is in fact at most poly-logarithmic in N, and at least logarithmic in N.

When $\alpha = 1$, the size of the largest component depends on the value of the parameter ν.

Theorem 1.6.2. *Assume that $\alpha = 1$. There exist constants $\frac{\pi}{8} \leq \nu_0 \leq \nu_1 \leq 20\pi$ such that the following hold:*

- *If $\nu < \nu_0$, then a.a.s. $|L_1| \leq \frac{N}{\log\log R}$.*

- *If $\nu > \nu_1$, then a.a.s. $|L_1| \geq N/610$.*

Chapter 3 deals with the proof of these two theorem.

Theorem 1.6.3. *Let $\alpha, \nu > 0$ be arbitrary. Then the following hold*

1. *If $\alpha > \frac{1}{2}$ then $G(N; \alpha, \nu)$ is a.a.s. disconnected.*

2. *If $\alpha < \frac{1}{2}$ then $G(N; \alpha, \nu)$ is a.a.s. connected.*

3. *If $\alpha = \frac{1}{2}$ then*

$$\lim_{N\to\infty} \mathbb{P}(G(N; \alpha, \nu) \text{ is connected }) = f(\nu),$$

where $f : (0, \infty) \to (0, 1]$ is a continuous function satisfying

(a) *$f(\nu) = 1$ for all $\nu \geq \pi$;*

(b) *$f(\nu)$ is strictly increasing for $0 < \nu < \pi$; and*

(c) $\lim\limits_{\nu\downarrow 0} f(\nu) = 0$.

This result highlights a strikingly different behaviour from all the other random graph models in the literature as far as we are aware, due the curious behaviour when $\alpha = \frac{1}{2}$. In that case, the limiting probability of connectedness is bounded away from zero and one for all $0 < \nu < \pi$, while it equals one for $\nu \geq \pi$.

The proof of this theorem can be found in Chapter 4.

Figure 1.2 shows a simulation of the function $f(\nu)$, sampling 1000 graphs on 50000 vertices for each value of ν, for $\nu = \{0, 0.2, \ldots, 3\}$. The source code for the simulation can be found in Appendix A.3.

The other property we are interested in are typical distances between two randomly chosen vertices. In Chapter 5, we show that $\mathcal{G}(N; \alpha, \nu)$ is *ultrasmall* when $\frac{1}{2} < \alpha < 1$, that is, when the degree distribution has a power law tail with exponent between 2 and 3. More specifically, we show that a.a.s. the graph distance between two randomly chosen vertices that belong to the same component is of order $\log\log N$. However, the diameter of the largest component of $\mathcal{G}(N; \alpha, \nu)$ grows at least logarithmically in N.

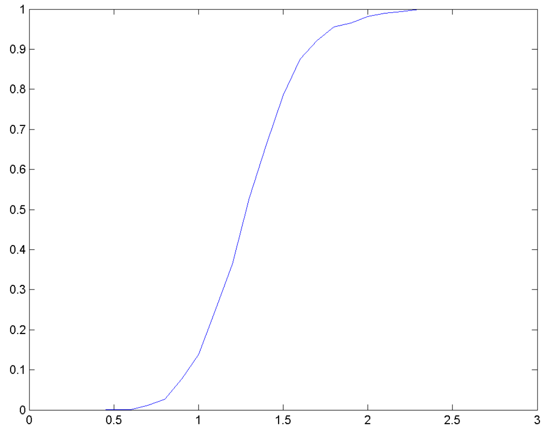

Figure 1.2: Approximation of $f(\nu)$, using $N = 50000$.

Friedrich and Krohmer showed that a.a.s. the the diameter This is a recent result of Friedrich and Krohmer [FK15], improving previous polylogarithmic bound of Kiwi and Mitsche [KM15].

For $\alpha > 1$, we show that a.a.s. $\mathcal{G}(N; \alpha, \nu)$ is *almost* ultrasmall: the graph distance between two randomly chosen vertices that belong to the same component is a.a.s. bounded by some polynomial of $\log\log N$. This range of α yields a power law degree distribution with exponent greater than 3.

Theorem 1.6.4. *Let $\zeta > 0$, and assume that $1/2 < \alpha < 1$. Let τ be such that $\tau^{-1} = \log\left(\frac{1}{2\alpha-1}\right)$. For $u, v \in \mathrm{V}_N$, a.a.s. if $d_G(u,v) < \infty$, then $\left|\frac{d_G(u,v)}{\log R} - 2\tau\right| < \zeta$.*

By Theorem 1.6.1, $\mathcal{G}(N; \alpha, \nu)$ has a giant component in this regime and therefore for any two distinct vertices u, v we have $d_G(u,v) < \infty$ with probability that is asymptotically bounded away from 0.

Unlike the Chung-Lu model, where an analogous result depends on the average degree, our result is formed independent on the choice of the parameter ν, which scales the average degree. The full result for the Chung-Lu model can be found in [VDH09].

The main idea behind the proof of Theorem 1.6.4 makes use of the existence of a very dense core that is formed by those vertices that have type at least $R/2$. We show that if two vertices are connected, then most likely they have short paths to the core which itself is a complete graph. These paths, which we call *exploding*, appear also in the Chung-Lu model [CL06, VDH09].

Our last result provides an upper bound on the typical distance between two connected vertices when $\alpha > 1$, when there is no giant component a.a.s. However, the largest component contains *polynomially many* vertices as there is a number of vertices of degree that scales polynomially in N. These components also form (almost) *ultrasmall* worlds.

Theorem 1.6.5. *Let $\alpha > 1$, $\varepsilon > 0$. A.a.s., there is a subset V' of vertices of $\mathcal{G}(N; \alpha, \nu)$ of size $(1-\varepsilon)N$ so that if $u, v \in V'$ and $d_G(u, v) < \infty$, then $d_G(u, v) \leq \log^{1+\varepsilon} \log N$.*

Figure 1.3 shows the behaviour of the graphs on $N = 1000$ vertices in different regimes, when α is 0.4, 0.75 and 1.2. The program is given in Appendix A.3.

The work presented in Chapters 3 and 4 is joint work with Nikolaos Fountoulakis and Tobias Müller [BFM15, BFM], Chapter 5 is joint work with Mohammed Amin Abdullah and Nikolaos Fountoulakis [ABF15].

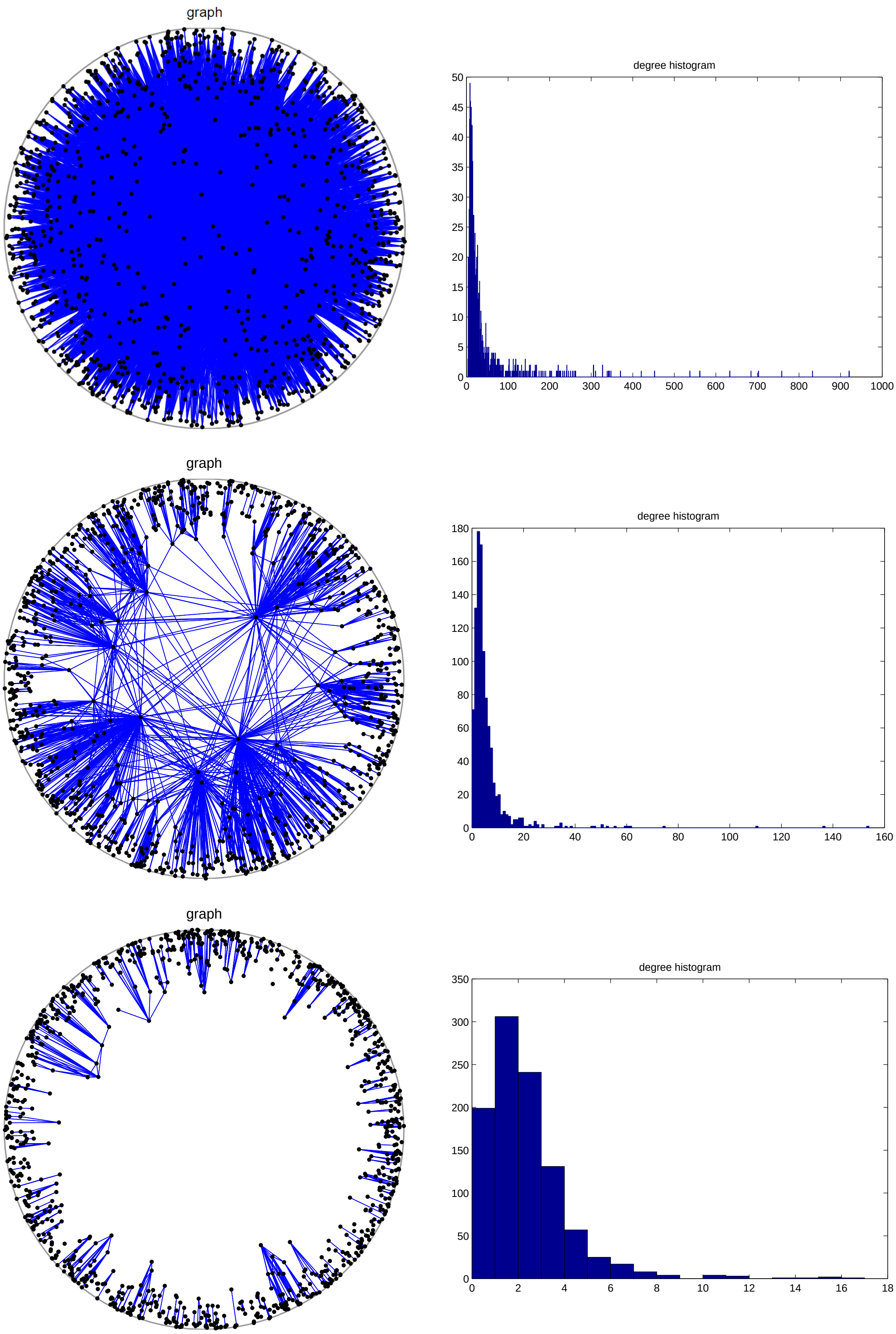

Figure 1.3: Simulations of the KPKVB-model with $N = 1000, \alpha = \{.4, .75, 1.2\}, \nu = 1$.

Chapter 2

Auxiliary results

In this chapter, we will state some basic results and definitions that we will use throughout this thesis.

2.1 Basic Facts about the KPKVB model

As discussed earlier, the radius of vertices primarily determines the role of a vertex. It is convenient to use the notion of the *type* of a vertex instead of the radius. For a vertex $u \in V_N$, its *type* t_u is defined to be equal to $R - r_u$ where r_u is the radius of u in $\mathcal{D}_R$. Similarly, a point $p \in \mathcal{D}_R$ of radius r_p has type $t_p = R - r_p$.

We start with a simple geometric fact. With O being the origin of $\mathcal{D}_R$, we say that a vertex v lies *above* some edge uw when v is inside the (hyperbolic) triangle Ouw, where uw is the geodesic path in $\mathcal{D}_R$ that joins u with w. Similarly, v lies *below* the edge uw, if v does not lie above uw but some radial projection of v towards O lies above uw.

Fact 2.1.1. *If the vertex u lies above the edge $u'u''$, then u is adjacent to u' and to u''. Moreover, the geodesic segments connecting u to u' and u'' lie entirely in the triangle $Ou'u''$.*

Proof. The hyperbolic triangle $Ou'u''$ has only sides of length at most R. The vertex u lies inside this triangle, so it has distance at most R from O, u' and u''. This is the case for any point w in the triangle $Ou'u''$. The geodesic from u to u' is entirely in the triangle, since otherwise it would have to cross one of the sides. A crossing point would therefore have two paths of minimum length to u', which is a contradiction. The same argument also works for the geodesic segment between u and u''. □

We will frequently use this in the form of the following weaker lemma, which states that every vertex in the neighbourhood of a given vertex u will still be connected to u when we increase t_u. Recall that $\theta_{u,v}$ is the relative angle between two vertices.

Lemma 2.1.2. *Let u, v and w be points in $\mathcal{D}_R$ with $\theta_{u,v} = 0$ and $t_u < t_v$. If $d(u,w) < R$ then $d(v,w) < R$.*

The lemma follows directly from Fact 2.1.1.

A very important identity we use is the *hyperbolic law of cosines* (see e.g. Anderson [And05]):

Fact 2.1.3.

$$\begin{aligned}\cosh(d(u,v)) = &\cosh(R-t_u)\cosh(R-t_v)\\ &- \sinh(R-t_u)\sinh(R-t_v)\cos(\theta_{u,v}).\end{aligned}$$

The combination of the facts that the cosine function is monotone decreasing on $[0,\pi]$, the hyperbolic cosine function is monotone increasing on $[0,\infty)$ and $\cosh(x) \geq \sinh(x)$ for all x with Fact 2.1.3 immediately gives us the following lemma:

Lemma 2.1.4. *If $u, v, w \in \mathcal{D}_R$ such that $t_v = t_w$ and $\theta_{u,v} < \theta_{u,w}$, then $d(u,v) < d(u,w)$.*

We use the combination of Lemmas 2.1.2 and 2.1.4 to "move" vertices to more convenient locations when this is helpful.

We note the following lemma, making it easier to work with the distribution of the types.

Lemma 2.1.5. *Uniformly for $0 \leq t < 0.99R$ we have*

$$\bar{\rho}(t) := \rho(R-t) = (1 + o_R(1))\alpha e^{-\alpha t}. \tag{2.1}$$

Proof. Using the pdf in Equation (1.1), we get

$$
\begin{aligned}
\rho(R-t) &= \alpha\frac{\sinh(\alpha(R-t))}{\cosh(\alpha R)-1} = \alpha\frac{e^{\alpha(R-t)}-e^{-\alpha(R+t)}}{2}\frac{2}{e^{\alpha R}+e^{-\alpha R}-2} \\
&= \alpha\frac{e^{\alpha(R-t)}-e^{-\alpha(R+t)}}{e^{\alpha R}+e^{-\alpha R}-2} = \alpha\frac{(1+o(1))e^{\alpha(R-t)}}{(1+o(1))e^{\alpha R}} \\
&= (1+o(1))\alpha e^{-\alpha t}.
\end{aligned}
$$

□

The following fact is an immediate consequence of the above from a first moment argument.

Corollary 2.1.6. *Let $\omega : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\omega(N) \to \infty$ as $N \to \infty$. The expected number of vertices of type at least $R/(2\alpha) + \omega(N)$ in $\mathcal{G}(N;\alpha,\nu)$ is $o(1)$. Hence, with probability $1-o(1)$ all vertices in V_N have type at most $\frac{1}{2\alpha}R+\omega(N)$.*

Thus, it suffices to consider vertices of type no larger than this bound.

The next lemma is key to our analysis throughout the thesis. It removes the need of dealing with hyperbolic distances, which are tedious to work out, but instead gives bounds for relative angles between vertices of given types. It reveals the part of $\mathcal{D}_R$ that includes all points within distance R of a given point as a drop, including the origin and (exponentially) growing thinner towards the periphery.

Lemma 2.1.7. *For any $\varepsilon > 0$ there exists an $N_0 > 0$ and a $c_0 > 0$ such that for any $N > N_0$ and $u, v \in \mathcal{D}_R$ with $t_u + t_v < R - c_0$ the following hold.*

- *If $\theta_{u,v} < 2(1-\varepsilon)\exp\left(\frac{1}{2}(t_u+t_v-R)\right)$, then $d(u,v) < R$.*
- *If $\theta_{u,v} > 2(1+\varepsilon)\exp\left(\frac{1}{2}(t_u+t_v-R)\right)$, then $d(u,v) > R$.*

Proof. We begin with the hyperbolic law of cosines (Fact 2.1.3):

$$
\begin{aligned}
\cosh(d(u,v)) = &\cosh(R-t_u)\cosh(R-t_v) \\
&- \sinh(R-t_u)\sinh(R-t_v)\cos(\theta_{u,v}).
\end{aligned}
$$

The right-hand side of the above becomes:

$$
\begin{aligned}
&\cosh(R-t_u)\cosh(R-t_v) - \sinh(R-t_u)\sinh(R-t_v)\cos(\theta_{u,v}) \\
&= \frac{e^{2R-(t_u+t_v)}}{4}\left[\left(1+e^{-2(R-t_u)}\right)\left(1+e^{-2(R-t_v)}\right)\right. \\
&\quad \left. - \left(1-e^{-2(R-t_u)}\right)\left(1-e^{-2(R-t_v)}\right)\cos(\theta_{u,v})\right] \\
&= \frac{e^{(2R-(t_u+t_v))}}{4}\left[1-\cos(\theta_{u,v}) + (1+\cos(\theta_{u,v}))\left(e^{-2(R-t_u)}+e^{-2(R-t_v)}\right)\right. \\
&\quad \left. + O\left(e^{-2(2R-(t_u+t_v))}\right)\right].
\end{aligned}
\tag{2.2}
$$

Therefore,

$$
\begin{aligned}
&\cosh(d(u,v)) \leq \\
&\frac{e^{(2R-(t_u+t_v))}}{4}\left[1-\cos(\theta_{u,v}) + 2\left(e^{-2(R-t_u)}+e^{-2(R-t_v)}\right) + O\left(e^{-2(2R-(t_u+t_v))}\right)\right].
\end{aligned}
$$

Since $t_u + t_v < R - c_0$, the last error term is $O(N^{-4})$. Also, it is a basic trigonometric identity that $1 - \cos(\theta_{u,v}) = 2\sin^2\left(\frac{\theta_{u,v}}{2}\right)$. The latter is at most $\frac{\theta_{u,v}^2}{2}$. Therefore, the upper bound on $\theta_{u,v}$ yields:

$$
\begin{aligned}
&\cosh(d(u,v)) \\
&\leq \frac{e^{2R-(t_u+t_v)}}{4}\left(\frac{\theta_{u,v}^2}{2} + 2\left(e^{-2(R-t_u)}+e^{-2(R-t_v)}\right) + O\left(\frac{1}{N^4}\right)\right) \\
&\leq \frac{e^{(2R-(t_u+t_v))}}{4}\left(2(1-\varepsilon)^2 e^{t_u+t_v-R} + 2\left(e^{-2(R-t_u)}+e^{-2(R-t_v)}\right)\right) + O(1) \\
&= (1-\varepsilon)^2\frac{e^R}{2} + \frac{1}{2}\left(e^{t_u-t_v}+e^{t_v-t_u}\right) + O(1) \\
&< (1-\varepsilon)^2\frac{e^R}{2} + \varepsilon\frac{e^R}{2} + O(1) < \frac{e^R}{2},
\end{aligned}
$$

for N sufficiently large and c_0 such that $e^{-c_0} < \frac{1}{2}\varepsilon$, since $t_u + t_v < R - c_0$ and $t_u, t_v \geq 0$. This implies that $t_u - t_v, t_v - t_u < R - c_0$ and, therefore, $\frac{1}{2}\left(e^{t_u-t_v} + e^{t_v-t_u}\right) < \frac{1}{2}\left(e^{R-c_0} + e^{R-c_0}\right) < \varepsilon\frac{e^R}{2}$. Also, since $\cosh(d(u,v)) > \frac{1}{2}e^{d(u,v)}$, it follows that $d(u,v) < R$.

To deduce the second part of the lemma, we consider a lower bound on (2.2) using the lower bound on $\theta_{u,v}$:

$$\begin{aligned}\cosh(d(u,v)) &\geq \frac{e^{2R-(t_u+t_v)}}{4}\left(1-\cos(\theta_{u,v})\right)+O(1)\\ &\geq \frac{e^{2R-(t_u+t_v)}}{4}\left(1-\cos\left(2(1+\varepsilon)e^{\frac{1}{2}(t_u+t_v-R)}\right)\right)+O(1).\end{aligned} \tag{2.3}$$

Using again that $1-\cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$ we deduce that

$$1-\cos\left(2(1+\varepsilon)e^{\frac{1}{2}(t_u+t_v-R)}\right) = 2\sin^2\left(\frac{1}{2}\ 4(1+\varepsilon)^2 e^{t_u+t_v-R}\right).$$

Since $t_u+t_v < R-c_0$, it follows that $t_u+t_v-R < -c_0$. So the latter is

$$\sin\left(\frac{1}{2}\ 4(1+\varepsilon)^2 e^{t_u+t_v-R}\right) > 2\left(1+\frac{\varepsilon}{2}\right)^2 e^{t_u+t_v-R},$$

for c_0 large enough, using the Taylor's expansion of the sine function around 0. Substituting this bound into (2.3) we have

$$\cosh(d(u,v)) \geq \left(1+\frac{\varepsilon}{2}\right)^2 \frac{e^R}{2} + O(1).$$

Thus, if $d(u,v) \leq R$, the left-hand side would be smaller than the right-hand side which would lead to a contradiction. □

We will define approximating areas of the ball of radius R around a given point u, motivated by Lemma 2.1.7. We call these bounding areas *inner* and *outer tube* of the point u.

Definition 2.1.8. *For a given point $u \in \mathcal{D}_R$ and for ε and N_0 as in Lemma 2.1.7 we call the sets*

- $T_u^- := \left\{v \in \mathcal{D}_R : \theta_{u,v} \leq 2(1-\varepsilon)\exp\left(\frac{1}{2}(t_u+t_v-R)\right)\right\}$ *the inner tube and*
- $T_u^+ := \left\{v \in \mathcal{D}_R : \theta_{u,v} \leq 2(1+\varepsilon)\exp\left(\frac{1}{2}(t_u+t_v-R)\right)\right\}$ *the outer tube*

of the point u.

Although by our definition there is no unique inner and outer tube, we will talk of the inner and outer tube. These should always be for suitably chosen ε and N_0 in the given context. Lemma 2.1.7 shows that, for sufficiently large graphs, all points in the inner tube of a typical vertex u (that is, a vertex of low type) are of distance at most R of u and all vertices of distance at most R of u are within the outer tube of u. We will use outer and inner tubes to derive stochastic bounds on the size of a component.

2.2 Poissonisation

It will sometimes be significantly easier to work in a setting where, instead of having exactly N random points, our vertex set consists of $\mathrm{Po}(N)$ points on $\mathcal{D}_R$, in the hyperbolic plane of curvature $-\alpha^2$. Two vertices/points are declared adjacent exactly as in $\mathcal{G}(N;\alpha,\nu)$. We denote the resulting graph by $\mathcal{P}(N;\alpha,\nu)$. More specifically, the vertex set consists of the points of a Poisson point process in $\mathcal{D}_R$ (see [Kin93]). In every measurable set $U \subseteq \mathcal{D}_R$, the number of points in U follows the Poisson distribution with parameter equal to $N\frac{\mathrm{Area}_\alpha(U)}{\mathrm{Area}_\alpha(\mathcal{D}_R)}$. Moreover, the numbers of points in any finite collection of pairwise disjoint measurable subsets of $\mathcal{D}_R$ are independent Poisson-distributed random variables.

We prove the following lemma that allows us to transfer results from the Poisson model into the $\mathcal{G}(N;\alpha,\nu)$ model. Let $\mathcal{A}_n$ denote a set of graphs on $\mathrm{V}_n := \{1,\dots,n\}$ that is closed under automorphisms. We call a family $\mathcal{A} = \{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of graphs *(vertex-) non-decreasing*, if $G - v \in \mathcal{A}_{n-1}$ for any[1] $v \in V(G)$ implies $G \in \mathcal{A}_n$. Similarly, we call the family *(vertex-) non-increasing*, if $G - v \notin \mathcal{A}_{n-1}$ for any $v \in V(G)$ implies $G \notin \mathcal{A}_n$.

Lemma 2.2.1. *Assume that $\alpha > 0$ is fixed. Let $\mathcal{A}$ be a (vertex-) non-increasing family of graphs. For N large enough we have $\mathbb{P}(\mathcal{G}(N;\alpha,\nu) \notin \mathcal{A}) < 4\mathbb{P}(\mathcal{P}(N;\alpha,\nu) \notin \mathcal{A})$. The same holds if $\mathcal{A}$ is (vertex-) non-decreasing.*

Proof. Denote by E_{Po} and E the events that $\mathcal{P}(N;\alpha,\nu) \notin \mathcal{A}$ and $\mathcal{G}(N;\alpha,\nu) \notin \mathcal{A}$,

[1] $G - v \in \mathcal{A}_{n-1}$ means that $G - v$ is isomorphic to a member of $\mathcal{A}_{n-1}$

respectively. We write

$$\begin{aligned}
\mathbb{P}(E_{Po}) &= \sum_{N'=0}^{\infty} \mathbb{P}(E_{Po}|Po(N) = N') \cdot \mathbb{P}(Po(N) = N') \\
&\geq \sum_{N'=N}^{\infty} \mathbb{P}(E_{Po}|Po(N) = N') \cdot \mathbb{P}(Po(N) = N') \\
&\geq \sum_{N'=N}^{\infty} \mathbb{P}(E_{Po}|Po(N) = N) \cdot \mathbb{P}(Po(N) = N'),
\end{aligned}$$

where we have used in the last line that we have $\mathbb{P}(E_{\mathrm{Po}}|\mathrm{Po}(N) = N') \geq \mathbb{P}(E_{\mathrm{Po}}|\mathrm{Po}(N) = N)$ for $N' \geq N$, since $\mathcal{A}$ is non-increasing. Let us also note that $\mathbb{P}(E_{\mathrm{Po}}|\mathrm{Po}(N) = N) = \mathbb{P}(E)$. Thus,

$$\begin{aligned}
\mathbb{P}(E_{\mathrm{Po}}) &\geq \sum_{N'=N}^{\infty} \mathbb{P}(E) \cdot \mathbb{P}(\mathrm{Po}(N) = N') \\
&= \mathbb{P}(E) \cdot \mathbb{P}(\mathrm{Po}(N) \geq N) \\
&> \frac{1}{4} \cdot \mathbb{P}(E),
\end{aligned}$$

where the last line holds for N large enough (by an application of, say, the central limit theorem). The second part of the lemma follows similarly, bounding the sum by taking only the terms where $N' \leq N$. □

This implies that if $\mathbb{P}(\mathcal{P}(N; \alpha, \nu) \notin \mathcal{A}) = o(1)$, then $\mathbb{P}(\mathcal{G}(N; \alpha, \nu) \notin \mathcal{A}) = o(1)$.

During some of our proofs, we will need to bound probabilities of events that are associated with a certain subset of vertices X, whose positions in $\mathcal{D}_R$ have been realised. For a certain measurable subset $U \subset \mathcal{D}_R$ which does not contain any vertex in X so that $\mathcal{D}_R \setminus U$ has positive Lebesgue measure, the vertices of the random graph $\mathcal{P}_{X,U}(N; \alpha, \nu)$ consist of X together with set of points of a Poisson process on $\mathcal{D}_R \setminus U$ with curvature $-\alpha^2$ with parameter $N - |X|$. Hence, this process "produces" $N - |X|$ vertices on average, thus giving N vertices in total on average. If we condition on the number of vertices of this Poisson process being N', then the resulting random graph is distributed as $\mathcal{G}(N'; \alpha, \nu)$ conditional on U *being empty* and X being located at particular positions.

Let $\mathcal{A}_X$ be a graph property associated with the set X. We call this *non-decreasing* if

$$\mathbb{P}_{\mathcal{P}_{X,U}(N;\alpha,\nu)}(\mathcal{A}_X \mid \mathrm{Po}(N-|X|)=N_1) \leq \mathbb{P}_{\mathcal{P}_{X,U}(N;\alpha,\nu)}(\mathcal{A}_X \mid \mathrm{Po}(N-|X|)=N_2),$$

whenever $N_1 \leq N_2$. If the opposite inequality holds, we call the property *non-increasing*. Note that $\mathbb{P}_{\mathcal{P}_{X,U}(N;\alpha,\nu)}(\mathcal{A}_X \mid \mathrm{Po}(N-|X|)=N')$ is the probability of $\mathcal{A}_X$ in the space $\mathcal{G}(N'+|X|;\alpha,\nu)$ conditional on X being at certain positions in $\mathcal{D}_R$ and U being empty – we denote this by $\mathcal{G}_{X,U}(N;\alpha,\nu)$. Hence, arguing as in the proof of the previous lemma we have

Lemma 2.2.2. *If $\mathcal{A}_X$ is either a non-decreasing or a non-increasing property that is associated with a certain set of vertices X, then*

$$\mathbb{P}_{\mathcal{P}_{X,U}(N;\alpha,\nu)}(\mathcal{A}_X) \geq \frac{1}{4}\mathbb{P}_{\mathcal{G}_{X,U}(N;\alpha,\nu)}(\mathcal{A}_X),$$

for any measurable $U \subset \mathcal{D}_R$ such that $X \cap U = \emptyset$ and $\mathcal{D}_R \setminus U$ has positive Lebesgue measure.

The following useful fact follows directly from the definition of the process, using the measure defined for the distribution of the points.

Fact 2.2.3. *Let A be a subset of $\mathcal{D}_R \setminus U$, for some measurable subset $U \subset \mathcal{D}_R$, and X be a set of vertices located in $\mathcal{D}_R$, such that $X \cap U, A \cap X = \emptyset$. Let N_A be the expected number of vertices in A, in $\mathcal{G}_{X,U}(N;\alpha,\nu)$, and denote by $\mathcal{E}_A$ the event that A is empty. We have*

$$\mathbb{P}_{\mathcal{P}_{X,U}(N;\alpha,\nu)}(\mathcal{E}_A) = \exp(-N_A).$$

2.3 The Breadth Exploration Process

To prove several of our results, we develop a technique reminiscent of branching processes. These can be used to prove equivalent results in other random graph models, such as the standard Erdős-Rényi model $G(N, p)$. A very important property of $G(N, p)$ is the independence of its edges. Branching process approximation heavily relies on this independence, vertices are exposed in steps and every edge of a current vertex to a non-exposed one is present with probability p, allowing for a precise stochastic analysis of the component of a given vertex in a breadth-first like fashion.

In the KPKVB model, this independence is not given, as the presence of edges depends on the position of the vertices. However, in $\mathcal{P}(N; \alpha, \nu)$, defined in the previous chapter, vertices in a subset of $\mathcal{D}_R$ can be exposed without changing the distribution of vertices in any disjoint subset of $\mathcal{D}_R$. Even though $\mathcal{G}(N; \alpha, \nu)$ does not have this property, as long as the subset of $\mathcal{D}_R$ and the number of vertices are relatively small compared to $\mathcal{D}_R$ and N, the distribution in disjoint subsets does not change a lot, as will be made precise when needed. We can use this property to bound the component a given vertex lies in by gradually exposing areas disjoint from everything exposed so far. The result is typically some *band* not containing any vertices and surrounding a given vertex v in a way that the component containing v cannot extend past the band, so all of the component has to be within the band. This then allows for the use of concentration arguments to show that the component in question must be small.

We use this process in one way or another in three parts of this thesis. In Chapter 3 we use it twice, to prove the non-existence of a "giant" component in Sections 3.1 and 3.3, while we use it in Chapter 5 to bound from above distances between vertices. The specific definition of the process varies depending on the circumstances and our aim, but it follows the following pattern:

- Given a vertex u, find its *most promising neighbour* in clockwise direction.
- If there are any, set this neighbour as u and repeat.
- Repeat the previous steps in the anti clockwise direction.

Firgure 2.1 depicts this process, with a red starting vertex, a series of most promising neighbors in the clockwise (green) and anti clockwise (blue) direction. The component

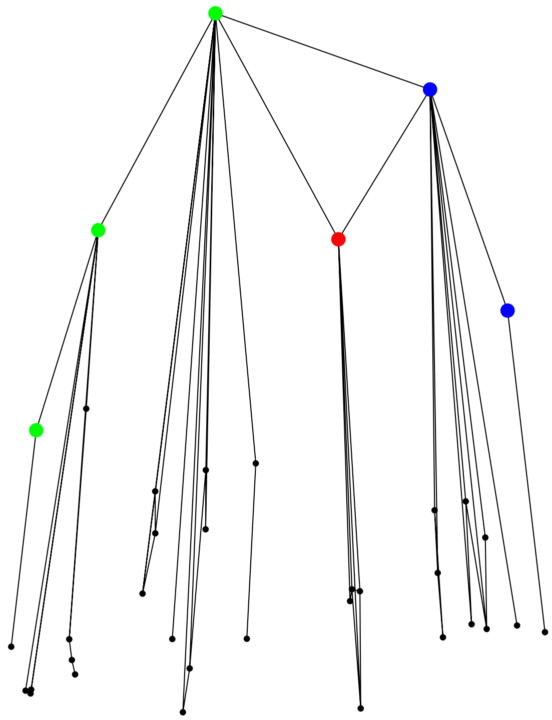

Figure 2.1: Example of a breadth exploration process.

was taken from simulated data with $N = 1000$, $\alpha = 1.2$ and $\nu = 1$.

We use the vague term of a *most promising neighbour* as this changes in the different versions. It might be the neighbour of highest type, but it could also be a point that does not correspond to a vertex, or we consider a set of points. In any case the choice is made in a way to get away from the original point as quickly as possible, which a high type point helps to accomplish.

The sequence of the vertices used defines some *bounding path*, a path that is not crossed by any edge and thus acts as a boundary for the component of the starting vertex.

In two of the versions we use, the process is simultaneously used in the clockwise and anticlockwise direction. This is required as the starting vertex v might not have a neighbour in anticlockwise direction, while one of its clockwise neighbours (that necessarily has to have higher type than v) has a neighbour that is in anticlockwise direction of v. We can only ignore this in the first occurrence of the process as we will assume that we start at the maximum possible type, preventing the "zigzagging" described earlier.

Chapter 3

The Emergence of a Giant Component

In this chapter we prove that a giant component appears when α crosses the value 1. Sections 3.1 and 3.2 deal with the subcritical case, i.e. nonexistence of a giant, and supercritical case, i.e. existence of a giant, respectively. In Section 3.3, we deal with the critical case when $\alpha = 1$.

3.1 Theorem 1.6.1: the subcritical case

Note that all vertices have type smaller than and asymptotically bounded away from $R/2$, by Corollary 2.1.6 and since $\alpha > 1$. We will consider a vertex u of type $\frac{1}{2\alpha}R+\omega(N)$ and analyse a breadth exploration process as motivated in Chapter 2.3, through which we will bound the *total angle* of the component $C(u)$ which contains u: We define

$$\Theta(u) := \max\left\{\theta_{v,w} \ : \ v, w \in C(u)\right\}.$$

This quantity represents the "width" of the component u belongs to.

We define a *bounding path*, which is a path on $\mathcal{D}_R$ that is not crossed by any edge. This is useful as we can use concentration results to bound the size of components on one side of the path. In particular, if a bounding path induces a partition of $\mathcal{D}_R$ into two parts, one of which covers an angle of at most $o(1)$, then a.a.s. any component in this part will be of sublinear size.

Definition 3.1.1. *We call a series of points $P = (p_1, p_2, \dots, p_m)$ in $\mathcal{D}_R$ a bounding path for $\mathcal{G}(N; \alpha, \nu)$, if the following hold:*

(i) The points p_1 and p_m are on the boundary of $\mathcal{D}_R$, i.e. their radius is R. Also, $\theta_{p_1,p_2} = \theta_{p_{m-1},p_m} = 0$.

(ii) For even i, we have $\theta_c(p_1, p_i) < \theta_c(p_1, p_{i+1})$ and $t_{p_i} = t_{p_{i+1}}$, while for odd i we have $\theta_c(p_1, p_i) = \theta_c(p_1, p_{i+1})$ and $t_{p_i} \neq t_{p_{i+1}}$.

(iii) Let $A \cup B$ be the partition of $\mathcal{D}_R$ incurred by P, using radial lines to connect vertices that only differ in type and arcs (lines of constant type) to connect vertices that only differ in angle. Let B be the part containing the origin and let A contain all points on the connections. There is no pair of adjacent vertices $a \in A$ and $b \in B$ in $\mathcal{G}(N; \alpha, \nu)$.

Note that (ii) ensures that P does not cross itself, so the path indeed partitions the disk into two parts and (iii) makes sense. Also, for $1 < i < m$ and any vertex v with $\theta_{v,p_i} = 0$ and $t_v < t_{p_i}$, the component of $\mathcal{G}(N; \alpha, \nu)$ that v belongs to covers an angle of at most $\theta_c(p_1, p_m)$. An example of a bounding path is given in Figure 3.1. The bounding path is the one resulting from using the breadth exploration process starting at the red vertex from Figure 2.1, with real vertices in blue and supporting points in red (these are not vertices in the graph).

We will now proceed with the definition of the *discrete breadth exploration process* that we will use to get a short bounding path. Note that $\frac{1}{2\alpha-1} < 1$ as $1 < 2\alpha - 1 \Leftrightarrow \alpha > 1$. We choose $\varepsilon > 0$ small enough so that $\frac{1}{\lambda} := 2\alpha - 1 - \varepsilon > 1$. Throughout this section we will need several small constants ε. We will assume that we choose one ε small enough for all these and require N to be large enough to satisfy everything. Given some constant $C > 0$, let i_0 be the minimum i such that $\lambda^i \left(\frac{1}{2\alpha} R + \omega(N)\right) < C$.

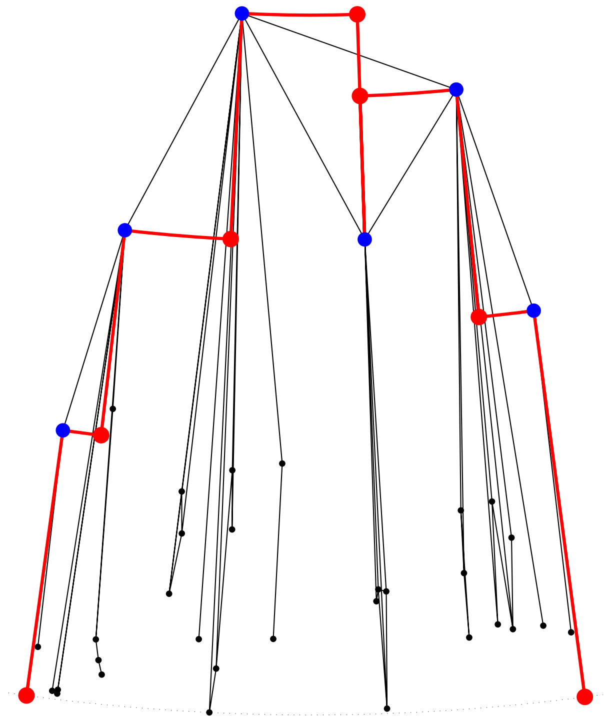

Figure 3.1: Example of a bounding path.

Note that $C_\lambda := \lambda^{i_0}\left(\frac{1}{2\alpha}R + \omega(N)\right) \geq \lambda C$. We partition the disk $\mathcal{D}_R$ into three bands:

$$\begin{aligned}\mathcal{B}_0 =& \{v \in \mathcal{D}_R : \frac{1}{2\alpha}R + \omega(N) < t_v \leq R\}\\ \mathcal{B}_{C_\lambda} =& \{v \in \mathcal{D}_R : C_\lambda < t_v \leq \frac{1}{2\alpha}R + \omega(N)\}\\ \mathcal{B}_{-} =& \{v \in \mathcal{D}_R : 0 \leq t_v \leq C_\lambda\}\end{aligned}$$

By Lemma 2.1.6 a.a.s. $\mathcal{B}_0$ does not contain any vertices. We define two phases for our random process, one on $\mathcal{B}_{C_\lambda}$ and one on $\mathcal{B}_{-}$. We start the process from a point $u \in \mathcal{D}_R$ with $0 < t_u \leq \frac{1}{2\alpha}R + \omega(N)$; in fact, due to Lemma 2.1.2 we may assume that $t_u = \frac{1}{2\alpha}R + \omega(N)$.

Phase I Letting $t_i := \lambda^i\left(\frac{1}{2\alpha}R + \omega(N)\right)$, we partition $\mathcal{B}_{C_\lambda}$ into i_0 bands

$$\mathcal{B}_{C_\lambda}^{(i)} = \{v \in \mathcal{D}_R : t_i < t_v \leq t_{i-1}\}$$

We know that there exists $i_u \in \{1, \ldots, i_0\}$ such that $u \in \mathcal{B}_{C_\lambda}^{(i_u)}$. We consider the *domain of attraction* around u:

$$\mathcal{A}_u := T_u^+ \cap \mathcal{B}_{C_\lambda} = \left\{ v : \theta_{u,v} \leq 2(1+\varepsilon)\nu \frac{e^{\frac{1}{2}(t_u+t_v)}}{N}, v \in \mathcal{B}_{C_\lambda} \right\}$$

and for $i = 1, \ldots, i_0$ we let $\mathcal{A}_u^{(i)}$ denote the set of points in $\mathcal{A}_u \cap \mathcal{B}_{C_\lambda}^{(i)}$ that are in the clockwise direction from u.

By Lemma 2.1.7, the domain of attraction $\mathcal{A}_u^{(i)}$ contains all points of the band $\mathcal{B}_{C_\lambda}^{(i)}$ that are within distance R in clockwise direction of the point u, but not every point in $\mathcal{A}_u^{(i)}$ must necessarily be within distance R of u. We define the first phase of the discrete breadth exploration process in the clockwise direction started at u as follows. Note that the auxiliary points defined in the process do not necessarily (in fact, with probability 1 they do not) correspond to vertices of the graph.

1. $v := u$ and $\Theta' := 0$; let i_v be such that $v \in \mathcal{B}_{C_\lambda}^{(i_v)}$;
2. let j_0 be the smallest i such that $\mathcal{A}_v^{(i)}$ contains a vertex;

   if such an index does not exist, then go to Phase II;

   if $j_0 \leq i_v$, the goto Step 5; (we then say that a *backward jump* occurs)
3. let $\hat{\Theta}_1 := 2\nu(1+\varepsilon)\frac{e^{1/2(t_v+t_{j_0-1})}}{N}$. Let w be the point of polar coordinates $(R - t_{j_0-1}, \theta_v - \hat{\Theta}_1)$.
4. go to Step 2, setting $v = w$ and $\Theta' := \Theta' + \hat{\Theta}_1$;
5. let v' be the point of polar coordinates $(R - t_0, \theta_v - 2\nu(1+\varepsilon)\frac{e^{\frac{1}{2}(t_v+t_0)}}{N})$; set $v := v'$;

Phase II

1. let v and Θ' have their final values after the execution of Phase I;
2. let $w \in \mathcal{D}_R$ be the point of type C_λ and $\theta_{v,w} = 2\nu(1+\varepsilon)\frac{e^{1/2(t_v+C_\lambda)}}{N}$ in the clockwise direction;

3. set $\Theta' := \Theta' + 2\nu(1+\varepsilon)\ \frac{e^{1/2(t_v+C_\lambda)}}{N}$ and let T_w^+ be the half-tube containing every point u that has relative angle at most $2\nu(1+\varepsilon)\ \frac{e^{1/2(t_u+C_\lambda)}}{N}$ with w in clockwise direction;

4. if T_w^+ is empty, then exit;

   else start the process again from Step 2 of Phase I with $\Theta' := \Theta' + 2\nu(1+\varepsilon)\ \frac{e^{1/2(t_0+C_\lambda)}}{N}$ and v of polar coordinates $(R - t_0, \theta_w - 2\nu(1+\varepsilon)\ \frac{e^{1/2(t_0+C_\lambda)}}{N})$.

Note that this process does not involve any points of type higher than t_0. Indeed, this is not necessary as by Lemma 2.1.6 a.a.s. all vertices in V_N have types no more than t_0.

We call a single execution of Steps 2-4 of Phase I a *round*. A maximal series of consecutive rounds is called a *cycle*. Thus, if at the end of a cycle a backward jump occurs, then Phase I proceeds to Step 5, initiating a cycle starting at a point of type t_0. This ensures that no matter where the backward jump takes place, vertices that are within distance R from the new root will be covered.

The set of rounds up to the end of Phase II is called an *epoch*. Hence, an epoch consists of repeated cycles, whose repetitions stop with an execution of Phase II. The *discrete breadth exploration process* starting at a vertex/point u is the process consisting of repeated epochs with the initial root v being the point of type t_0 and relative angle with respect to u that is equal to 0. (Thus, in fact, the process does not start from u but at the "image" of u that has type t_0.)

Remember that $\Theta(u)$ is the maximum relative angle between any two vertices in the component that contains u. We prove the following lemma:

Lemma 3.1.2. *For any vertex $u \in \mathrm{V}_N$ of type less than t_0, if Θ' denotes the maximum of the angles gained during the breadth exploration process started at u', a point of type $t_{u'}$ and relative angle 0 to u, in the clockwise and the anticlockwise direction, then the process yields a bounding path and thus $\Theta(u) \leq 2 \cdot \Theta'$.*

Proof. Note that, by Lemma 2.1.2, raising the type of u cannot make the component smaller and is necessary as otherwise the clockwise and anticlockwise directions might influence each other. Using the discrete breadth exploration process in the clockwise direction, we get a series of root points - these are the vertices in the beginning of Phase I. Let $u_1, \ldots, u_m$ be the part of this series that corresponds to the last cycle, i.e. there is no backwards jump within $u_1, \ldots, u_m$ and $u_1 = u'$ or there was a backwards jump right before u_1. Let $\hat{u}_i$ be the radial projection of the point u_i to type $t_{u_{i+1}}$. The series $u_1, \hat{u}_1, u_2, \hat{u}_2, \ldots, \hat{u}_{m-1}, u_m$ thus always alters between changing the type and relative angle, as required in condition (ii). Similarly, in anticlockwise direction, we get the series $u'_1, \hat{u}'_1, u'_2, \hat{u}'_2, \ldots, \hat{u}'_{\ell-1}, u'_\ell$. Letting $\hat{u}_m$ and $\hat{u}'_\ell$ be the radial projections of u_m and u'_ℓ to the boundary of $\mathcal{D}_R$, we get the path $P = (\hat{u}'_\ell, u'_\ell, \hat{u}'_{\ell-1}, \ldots, u'_1, u_1, \ldots, \hat{u}_{m-1}, u_m, \hat{u}_m)$. If the discrete breadth exploration process only uses a total angle that is $o(1)$, which is the case a.a.s., then (i) and (ii) are naturally true if $u_1 \neq u'_1$. If $u_1 = u'_1$ (implying $u_1 = u' = u'_1$), almost surely (with probability 1) we can push u_1 in clockwise direction by some small amount (angle $o(\frac{1}{N})$) to fix this problem without causing further problems elsewhere (i.e. all the adjacencies of u_1 stay the same).

To prove that P is a bounding path for $\mathcal{G}(N; \alpha, \nu)$ we need to show that there is no pair of vertices (v, w) such that $v \in A$, $w \in B$ and $v \sim w$. Assume for a contradiction that there is such an edge vw. Without loss of generality we only consider the series $P_1 = (u_1, \hat{u}_1, u_2, \hat{u}_2, \ldots, \hat{u}_{m-1}, u_m)$. If $\theta_c(p_1, w) \leq \theta_c(p_1, u_m)$, then there are two consecutive root points, u_i and u_{i+1} such that $\theta_c(p_1, u_i) < \theta_c(p_1, w) \leq \theta_c(p_1, u_{i+1})$. By Lemmas 2.1.2 and 2.1.4 and since $t_{u_{i+1}} < t_w$, we have $d(u_i, w) < R$, a contradiction to the choice of u_{i+1} as the next root vertex.

Now assume that $\theta_c(p_1, w) > \theta_c(p_1, u_m)$. By Lemma 2.1.2 and as $v \in A$, there is an $i \geq 1$ such that $\theta_c(p_1, \hat{u}_{i-1}) < \theta_c(p_1, v) \leq \theta_c(p_1, u_i)$(where, for convenience, $\hat{u}_0 = u'$). But as $v \in A$, we have $t_v \leq t_{u_i}$, so by Lemmas 2.1.2 and 2.1.4 we have that u_i is adjacent to w. By the choice of u_{i+1} in the discrete breadth exploration process, we thus have $\theta_c(p_1, w) \leq \theta_c(p_1, u_{i+1}) \leq \theta_c(p_1, u_m)$ and $t_w < t_{u_{i+1}}$, so $w \in A$, a contradiction. Note that $i < m$ since as u_i is adjacent to w the discrete breadth exploration process cannot have stopped at u_i.

So using the discrete breadth exploration process twice we indeed find a bounding path. In particular, the angle gained in both direction gives a slice of the disk that contains the entire component of u. $\square$

We now want to bound from above the angle that can be gained during the execution of the process.

Lemma 3.1.3. *Let $u \in \mathcal{D}_R$ be a point having $t_u = t_0$. If the discrete breadth exploration process starts at point u, then by the end of it $\Theta' \leq R^2 \log^3 RN^{1/\alpha-1}$ with probability $1 - o\left(N^{1/\alpha-1}\right)$.*

Proof. Let us consider the discrete breadth exploration process started at a vertex u having type $t_0 = \frac{1}{2\alpha}R + \omega(N)$. For an $\varepsilon > 0$ we let T_ε denote the first round at the end of which $\Theta' \geq \varepsilon$ if there is such a round, otherwise $T_\varepsilon = \infty$. We also denote by $u_0(t)$ the root vertex at the beginning of the tth round and let $i_{u_0} \geq 0$ denote the index of the band this vertex belongs to. We will first bound from below the probability that the exploration process does not backtrack during the tth round. Let $B_t^{(i_{u_0})}$ be the indicator random variable that is equal to 1 if and only if backtracking *does* occur during the tth round assuming that the root vertex is in i_{u_0}.

Claim 3.1.4. *For $\varepsilon \in (0, 2\pi)$, let $t < T_\varepsilon$. There exists a constant $K = K(\alpha, \nu) > 0$ such that for any N that is sufficiently large we have*

$$\Pr\left[B_t^{(i_{u_0})} = 0\right] > \exp\left(-Ke^{-\frac{\varepsilon}{2}t_{i_{u_0}}}\right).$$

Proof of Claim 3.1.4. Let us write $u_0 = u_0(t)$. For $t < T_\varepsilon$ we give a stochastic upper bound on the number of vertices that belong to $\cup_{j=0}^{i_{u_0}} \mathcal{A}_{u_0}^{(j)}$. Hence, we will be able to give a lower bound on the probability that this region is empty. In other words, we will bound from below the probability that no backtracking occurs during the tth round. Let N_t denote the number of vertices that have not been exposed at the beginning of the tth round. Using Lemma 2.1.5, the probability that one of them will belong to $\cup_{j=0}^{i_{u_0}}\mathcal{A}_{u_0}^{(j)}$ is bounded from above by

$$\begin{aligned}
&\frac{2\nu(1+\varepsilon)\alpha}{2\pi-\Theta'}\int_{t_{i_{u_0}}}^{t_0}\frac{e^{\frac{1}{2}(t_{u_0}+s)}}{N}e^{-\alpha s}ds \leq \frac{2\nu(1+\varepsilon)\alpha}{2\pi-\varepsilon}\int_{t_{i_{u_0}}}^{t_0}\frac{e^{\frac{1}{2}(t_{u_0}+s)}}{N}e^{-\alpha s}ds \\
&\leq \frac{4\nu(1+\varepsilon)\alpha}{(2\pi-\varepsilon)(2\alpha-1)}\frac{e^{t_{u_0}/2}}{N}\ e^{(1/2-\alpha)t_{i_{u_0}}} \\
&\leq \frac{4\nu(1+\varepsilon)\alpha}{(2\pi-\varepsilon)(2\alpha-1)}\frac{e^{\frac{1}{2}(t_{i_{u_0}-1}+t_{i_{u_0}})-\alpha t_{i_{u_0}}}}{N} =: p_t^{(i_{u_0})}.
\end{aligned}$$

Hence, the number of vertices which during round t will fall into $\cup_{j=0}^{i_{u_0}}\mathcal{A}_{u_0}^{(j)}$ is binomially distributed with parameters $N_t, p_t^{(i_{u_0})}$. In turn, this is stochastically bounded from above by a binomially distributed random variable with parameters $N, p_t^{(i_{u_0})}$. Note also that if the number of vertices that fall into $\cup_{j=0}^{i_{u_0}}\mathcal{A}_{u_0}^{(j)}$ is positive, then backtracking occurs. Hence, the probability of not backtracking during round t is at least $\Pr\Big[\mathsf{Bin}(N, p_t^{(i_{u_0})}) = 0\Big]$.

Setting $K' = \frac{4\nu(1+\varepsilon)\alpha}{(2\pi-\varepsilon)(2\alpha-1)}$, we now obtain an asymptotic estimate on this probability:

$$\begin{aligned}
\Pr\Big[\mathsf{Bin}(N, p_t^{(i_{u_0})}) = 0\Big] &= \left(1 - p_t^{(i_{u_0})}\right)^N \\
&= \exp\left(-K' e^{\frac{1}{2}(t_{i_{u_0}-1}+t_{i_{u_0}})-\alpha t_{i_{u_0}}}(1+o(1))\right).
\end{aligned}$$

But recall that $t_{i_{u_0}-1} = \frac{1}{\lambda}t_{i_{u_0}} = (2\alpha-1-\varepsilon)\,t_{i_{u_0}}$, whereby

$$\begin{aligned}
\frac{1}{2}(t_{i_{u_0}-1}+t_{i_{u_0}}) &= \frac{1}{2}\left(2\alpha-1-\varepsilon+1\right)t_{i_{u_0}} = \frac{1}{2}\left(2\alpha-1-\varepsilon+1\right)t_{i_{u_0}} \\
&= \left(-\frac{\varepsilon}{2}+\alpha\right)t_{i_{u_0}}.
\end{aligned}$$

This shows that $p_t^{(i_{u_0})} = O(\frac{1}{N})$, justifying the above exponential approximation. Hence, we obtain

$$\begin{aligned}
\Pr\Big[\mathsf{Bin}(N, p_t^{(i_{u_0})}) = 0\Big] = \left(1 - p_t^{(i_{u_0})}\right)^N &= \exp\left(-(1+o(1))K' e^{-\frac{\varepsilon}{2}t_{i_{u_0}}}\right) \\
&> \exp\left(-2K' e^{-\frac{\varepsilon}{2}t_{i_{u_0}}}\right),
\end{aligned}$$

for any N sufficiently large, uniformly over all possible values of i_{u_0}. (The latter is the case since always $t_{i_{u_0}} < R/2$.) Taking $K = 2K'$, the claim now follows. □

Now, observe that the above claim implies that the probability of no backtracking at a certain round can become very close to 1. Indeed, note that $t_{u_0} \geq \lambda C$ and, therefore, the exponent on the right-hand side of the bound obtained in Claim 3.1.4 can be made as close to 0 as we want, provided we choose C large enough. Moreover, if t_{u_0} is bounded from below by a function of N that increases as $N \to \infty$, then the probability of no backtracking is in fact $1 - o(1)$. These observations are key to the deduction of the first part of the lemma.

We first show that provided that Θ' is much less than ε, the number of cycles within an epoch is essentially stochastically dominated by a geometrically distributed random variable that has probability of success $1 - \varepsilon$, provided that the parameter $C = C(\varepsilon)$ is large enough. Suppose that an epoch starts with $\Theta' \leq g(N)$ where $g(N) = o(1)$.

Recall that a cycle starts at a vertex that has type $t_0 = \frac{1}{2\alpha}R + \omega(N)$. Let T_{C_λ} denote the random variable that is the length of a cycle. We say that a cycle is *successful* if it exits to Phase II. Note that a cycle is successful, that is, no backtracking occurs, if and only if $B_t^{(i_{u_0})} = 0$, for all $t \leq T_{C_\lambda}$.

We will bound the probability that, conditional on $\Theta' \leq g(N)$ at the beginning of the epoch, the number of cycles is at least R. In particular, we will show that for every ε there exists a C such that this probability is at most ε^{R-1}.

Claim 3.1.5. *Let $g(N) = o(1)$. For every $\varepsilon > 0$ there exists a $C = C(\varepsilon)$ such that for any N sufficiently large, conditional on $\Theta' \leq g(N)$ at the beginning of an epoch, with probability at least $1 - \varepsilon^{R-1}$ the total angle gained during the epoch is at most $2R \log^2 R N^{1/\alpha - 1}$.*

Proof. To bound this probability, we will repeatedly apply Claim 3.1.4. However, in order to do this we need to ensure that Θ' does not exceed ε whenever at most R cycles have been executed.

Hence, we first need to give an upper bound on the angle that is gained during the execution of a cycle. If $\Theta'_{T_{C_\lambda}}$ denotes this angle, then, by Lemma 2.1.7,

$$\Theta'_{T_C} < 2\nu(1+\varepsilon)\sum_{i=0}^{i_0} \frac{e^{\frac{1}{2}(t_i+t_{i+1})}}{N}.$$

But for all i we have $t_i < \frac{1}{2\alpha}R + \omega(N)$. Hence, $t_i + t_{i+1} < \frac{1}{\alpha}\ R + 2\omega(N)$, whereby

$$\frac{e^{\frac{1}{2}(t_i+t_{i+1})}}{N} < \frac{e^{\frac{1}{2}R\ \frac{1}{\alpha}+\omega(N)}}{N} = e^{\omega(N)}\ N^{\frac{1}{\alpha}-1}.$$

Using this with $\omega(N) = \log\log^{\frac{1}{2}} R$, the above sum can be further bounded from above by

$$\Theta'_{T_C} < 2\nu(1+\varepsilon)(i_0+1)e^{\log\log^{\frac{1}{2}} R}\ N^{\frac{1}{\alpha}-1} \stackrel{i_0=O(\log R)}{\leq} \log^2 R\frac{N^{1/\alpha}}{N}, \tag{3.1}$$

if N is large enough. Therefore, after $r \leq R$ cycles the angle gained will be at most $R\log^2 R\frac{N^{1/\alpha}}{N} < \varepsilon$, for any N that is sufficiently large. Note also that this quantity bounds the total angle that is gained during an epoch consisting of at most R cycles.

Hence, applying Bayes' rule repeatedly, Claim 3.1.4 implies that

$$\Pr\Big[\,B_t^{(i_{u_0})} = 0,\ \forall t < T_{C_\lambda}\Big] \geq \prod_{i=0}^{i_0} \exp\left(-Ke^{-\frac{\varepsilon}{2}t_i}\right). \tag{3.2}$$

Let $a_i := e^{-\frac{\varepsilon}{2}t_i}$ and note that since $t_i = \lambda^i t_0$ we have for $i \leq i_0$

$$\begin{aligned}\frac{a_{i-1}}{a_i} &= e^{-\frac{\varepsilon}{2}t_0\left(\lambda^{i-1}-\lambda^i\right)} = e^{-\frac{\varepsilon}{2}t_0\lambda^i\left(\frac{1}{\lambda}-1\right)} \leq e^{-\frac{\varepsilon}{2}t_0\lambda^{i_0}\left(\frac{1}{\lambda}-1\right)}\\ &\leq e^{-\frac{\varepsilon}{2}\lambda C\left(\frac{1}{\lambda}-1\right)}.\end{aligned}$$

Thus, if C is large enough, then

$$\sum_{i=0}^{i_0} e^{-\frac{\varepsilon}{2}t_i} < e^{-\frac{\varepsilon}{2}t_{i_0}} \sum_{i=0}^{\infty} e^{-\frac{\varepsilon}{2}i\cdot C(1-\lambda)} \stackrel{t_{i_0}>\lambda C}{<} 2e^{-\frac{\varepsilon}{2}\lambda C}.$$

Substituting this bound into the right-hand side of (3.2) we obtain

$$\Pr\Big[B_t^{(i_{u_0})} = 0,\ \forall t \leq T_{C_\lambda} \Big] \geq \exp\left(-2Ke^{-\frac{\varepsilon}{2}\lambda C}\right) > 1 - 2Ke^{-\frac{\varepsilon}{2}\lambda C} > 1 - \varepsilon, \qquad (3.3)$$

choosing C large enough so that the last inequality holds. Hence, the probability that backtracking occurs before T_{C_λ} is at most ε, for $C = C(\varepsilon)$ that is sufficiently large. In other words, the probability that the cycle is not successful conditional on $\Theta' \leq g(N)$ at the beginning of the epoch is at most ε. Therefore, the conditional probability of having R cycles during the epoch is at most ε^{R-1}.

As we pointed out above the total angle that was gained above during the execution of the R cycles is no more than $R \log^2 RN^{1/\alpha - 1}$. During Phase II, the angle gained is at most

$$4\nu(1+\varepsilon)\frac{e^{\frac{1}{2}(t_0 + C_\lambda)}}{N} \leq 4\nu(1+\varepsilon)e^{\frac{1}{2}(C_\lambda + \omega(N))}\frac{e^{\frac{1}{2\alpha}\frac{1}{2}R}}{N} = o\left(N^{\frac{1}{\alpha}-1}\right).$$

Hence, an epoch having at most R cycles adds at most $2R\log^2 RN^{1/\alpha-1}$ to Θ', provided that N is sufficiently large. □

Now, we will show that as long as Θ' has not grown too much, the probability that an epoch is the final one is asymptotically bounded away from 0. To see this, we will bound from above the probability that T_v^+, that was defined in Step 3 of Phase II, contains at least one vertex conditional on $\Theta' \leq \varepsilon$. In particular, conditional on this, the probability that a given vertex whose exact position in $\mathcal{D}_R$ has not been exposed yet belongs to T_v^+ is at most

$$\frac{2\nu\alpha(1+\varepsilon)}{2\pi - \varepsilon}\int_0^{t_0} \frac{e^{\frac{1}{2}(C_\lambda + t)}}{N} e^{-\alpha t} dt > \frac{2\nu\alpha(1+\varepsilon)}{2\pi-\varepsilon}\frac{e^{C_\lambda/2}}{N}\int_0^\infty e^{\left(\frac{1}{2}-\alpha\right)t}dt$$
$$\stackrel{1/\alpha<2}{=} \frac{4\nu\alpha(1+\varepsilon)}{(2\alpha-1)(2\pi-\varepsilon)}\frac{e^{C_\lambda/2}}{N} =: p_{II} = O(\frac{1}{N}).$$

Under the above conditioning, the number of vertices that belong to T_v^+ is stochastically bounded from above by a binomially distributed random variable with parameters

N, p_{II} Hence, the probability that T_v^+ is empty conditional on $\Theta' \leq \varepsilon$ is at least

$$\begin{aligned}(1-p_{II})^N &= \exp\left(-\frac{4\nu(1+\varepsilon)}{(2\alpha-1)(2\pi-\varepsilon)}\ e^{C_\lambda/2}(1+o(1))\right)\\ &\geq \exp\left(-\frac{4\nu(1+\varepsilon)}{\pi(2\alpha-1)}\ e^{C_\lambda/2}\right) =: \delta,\end{aligned}$$

provided that $\varepsilon < \pi$ and N is sufficiently large.

Now, we set $E := \lfloor -\frac{1-1/\alpha}{\ln\delta}\ R\rfloor$. We will finish the proof by showing that the probability that less than E epochs take place each having at most R cycles is $1 - o\left(N^{1/\alpha-1}\right)$. This together with Claim 3.1.5 imply that with this probability the total angle gained during the process is at most $2ER\log^2 RN^{1/\alpha-1}$.

Indeed, the probability of having E epochs each one having at most R cycles is at least

$$\delta^E \geq \delta^{-\frac{1-1/\alpha}{\ln\delta}\ R} = \exp\left((1/\alpha-1)\ R\right) = O(N^{2(1/\alpha-1)}),$$

and the latter is $o\left(N^{1/\alpha-1}\right)$. Also, arguing as in the proof of Claim 3.1.5, we deduce that the probability that there exists one among the first E cycles having more than R cycles is at most $E\varepsilon^R = o(N^{1/\alpha-1})$, provided that ε is chosen small enough. □

The above lemma together with Lemma 3.1.2 imply that

Lemma 3.1.6. *For any $u \in \mathrm{V}_N$ we have $\Theta(u) \leq R^2\log^3 RN^{1/\alpha-1}$ with probability $1-o\left(N^{1/\alpha-1}\right)$.*

We will now deduce the first part of Theorem 1.6.1 from Lemma 3.1.6. Let $\mathcal{B}$ denote the set of vertices u of type at most t_0 for which $\Theta(u) > 2R^2\log^3 RN^{1/\alpha-1}$ - we call these vertices *bad*. Thus, Lemma 3.1.6 implies that

$$\mathbb{E}\left[|\mathcal{B}|\right] = o(N^{1/\alpha}).$$

Markov's inequality in turn implies that for any $\delta > 0$ we have that with probability $1-o(1)$

$$|\mathcal{B}| < \delta N^{1/\alpha}. \tag{3.4}$$

Let us assume now that $\mathcal{G}(N;\alpha,\nu)$ has a component C that is of order greater than $8R^2\log^3 RN^{1/\alpha}$. Hence, on the event (3.4), there is at least one (in fact, many) vertex $u \in C$ in this component that is *not* bad.

A *sector* of $\mathcal{D}_R$ is the area between two radii of $\mathcal{D}_R$ of relative angle which is less than π - we call this angle the *angle of the sector*. Hence, since u is not bad, it turns out that there is a sector of angle at most $2R^2\log^3 RN^{1/\alpha-1}$ which contains at least $4R^2\log^3 RN^{1/\alpha}$ vertices (in fact, our assumption implies that it contains almost twice as many vertices as this). But the next lemma shows that this is not the case with probability $1-o(1)$ and the first part of Theorem 1.6.1 follows.

Lemma 3.1.7. *Let $\theta : \mathbb{N} \to \mathbb{R}$ be a non-negative function such that $\theta(N) = o(1)$ but $\frac{N\theta(N)}{\ln(1/\theta(N))} \to \infty$ as $N \to \infty$. Then a.a.s. there is no sector of angle $\theta(N)$ that contains at least $2N\theta(N)$ vertices.*

Proof. Consider a partition $\mathcal{P}$ of $\mathcal{D}_R$ into $2\pi/\theta(N)$ sectors of angle $\theta(N)$. If $\mathcal{D}_R$ contains a sector as in the statement of the lemma, then one of the sectors in $\mathcal{P}$ must contain at least $N\theta(N)$ vertices. Now, note that the number of vertices is a sector $\sigma \in \mathcal{P}$, which we denote by N_s, is binomially distributed with parameters N and $\theta(N)/2\pi$. Hence, $\mathbb{E}(N_\sigma) = \frac{1}{2\pi}\, N\theta(N)$ and since $N\theta(N) \to \infty$ and applying a Chernoff-type bound we deduce that

$$\Pr\Big[\, N_\sigma > N\theta(N) \,\Big] = \exp\left(-\Omega(N\theta(N))\right).$$

Therefore, using Markov's inequality we obtain:

$$\Pr\Big[\,\exists\, \sigma \in \mathcal{P} \;:\; N_\sigma > N\theta(N)\,\Big] \leq \frac{2\pi}{\theta(N)} \exp\left(-\Omega(N\theta(N))\right) = o(1),$$

which concludes the proof of the lemma. $\square$

3.2 Theorem 1.6.1: the supercritical case

In this section, we will show the second part of Theorem 1.6.1. Namely, we shall assume that $\alpha < 1$ and with $|L_1|$ denoting the size of a largest component of $\mathcal{G}(N;\alpha,\nu)$, we will show that there exists a $c = c(\alpha, \nu)$ such that a.a.s. $|L_1| > cN$.

3.2.1 Proof overview

We will consider a set of homocentric bands in $\mathcal{D}_R$. The innermost band consists of those vertices of type at least $R/2$. Note that the subgraph of $\mathcal{G}(N; \alpha, \nu)$ that is induced by the vertices which belong to this part of $\mathcal{D}_R$ is a clique. This follows from the triangle inequality, which implies that the distance between any two vertices there is at most R. The remaining bands are determined by a sequence of numbers t_i, with $t_0 = R/2$ and t_i defined by the following recursion:

$$t_i - 2\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i\right) = \lambda t_{i-1}, \tag{3.5}$$

if $0 < t_i < t_{i-1}$. where now $\lambda := 2\left(\alpha - \frac{1}{2}\right)$ – we assume that $\alpha > \frac{1}{2}$. The bands are now as follows:

$$\mathcal{B}_0 = \{v \in \mathcal{D}_R : R/2 < t_v \leq R\} \text{ and } \mathcal{B}_i = \{v \in \mathcal{D}_R : t_i < t_v \leq t_{i-1}\}.$$

We shall assume that $i < T$, where $T = T(\alpha, \nu, \varepsilon)$ and ε is a positive real number which we will assume to be small enough for the purposes of our calculations. We will determine T in Subsection 3.2.2. Observe that (3.5) implies that provided that $t_i > \nu(1-\varepsilon)^4/(4\pi)$,

$$t_i > \lambda^i t_0 \text{ for } i > 0. \tag{3.6}$$

We denote by $\mathcal{N}_i$ the set of vertices which belong to the ith band, for $i \geq 0$, and let N_i denote its size. Furthermore, for $i > 0$ we denote by $\mathcal{N}_i'$ the set of vertices in $\mathcal{B}_i$ that have at least one neighbour in $\mathcal{N}_{i-1}'$ – here we set $\mathcal{N}_0' = \mathcal{N}_0$. We say that these belong to the *active area* of $\mathcal{B}_i$. This definition together with the fact that a clique is formed in $\mathcal{N}_0'$ imply that the graph induced by $\bigcup_{i=0}^T \mathcal{N}_i'$ is connected and contains $\sum_{i=0}^T |\mathcal{N}_i'|$ vertices. Our aim is to show that a.a.s. this quantity is linear in N. We let $N_i' = |\mathcal{N}_i'|$.

More specifically, we show that the number of vertices in $\mathcal{N}_i'$ stochastically dominates the number of vertices in a subset of $\mathcal{B}_i$ that has arc length Θ_i. This makes working with sizes a lot easier, as implications for the size of $\mathcal{N}_i'$ can be deduced from the angle Θ_{i-1}. In particular, for any fixed i, with probability $1 - o\left(\frac{1}{\ln N}\right)$

$$N_{i-1}' \geq N_{i-1} \frac{\Theta_{i-1}}{2\pi}(1-\varepsilon). \tag{3.7}$$

The proof of this can be found in Section 3.2.4. Next we argue (cf. Section 3.2.4) that conditional on N_{i-1}' as above and $\Theta_{i-1} > \pi$ with high probability Θ_i is at least a certain fraction of Θ_{i-1}.

Lemma 3.2.1. *Conditional on $N_{i-1} \in (1 \pm \varepsilon)\mathbb{E}\left[N_{i-1}\right]$, on $\Theta_{i-1} > \pi$ as well as on N_{i-1}' satisfying (3.7) with probability $1 - o\left(\frac{1}{\ln N}\right)$ we have*

$$\Theta_i \geq \Theta_{i-1}\left(1 - e^{-\gamma t_i}\right), \tag{3.8}$$

for some constant $\gamma = \gamma(\alpha, \nu, \varepsilon) > 0$, uniformly for $i = 1, \ldots, T$.

Note that we take $\Theta_0 := 2\pi$.

To derive the stochastic domination we will assume that the following conditions hold:

- any vertex of type t with $t_i \geq R/2$ is of type $R/2$;
- any vertex of type t with $t_i < t \leq t_{i-1}$ is of type t_i.

Lemma 2.1.2 ensures that for a vertex $v \in \mathcal{B}_{i-1}$ the area consisting of all points that belong to $\mathcal{B}_i$ and have distance R from v becomes smaller, if the type of v within the bounds of $\mathcal{B}_{i-1}$ decreases. Now, using the first part of Lemma 2.1.7, we can use the inner tubes to obtain a further lower bound on this area. In particular, we will consider only vertices that fall within the inner tube of $v \in \mathcal{N}_{i-1}'$ assuming that the type of v is t_{i-1} and deduce a stochastic lower bound on the size of $\mathcal{N}_i'$.

Using concentration arguments we will show that a.a.s. $N_i' \geq \frac{1}{2}\, N \frac{\Theta_i}{2\pi}\, (e^{-\alpha t_i} - e^{-\alpha t_{i-1}})$. Hence, if T is such that

$$\prod_{i=0}^{T} \left(1 - e^{-\gamma t_i}\right) > \frac{1}{2}, \tag{3.9}$$

and for some $C > 0$

$$t_T \leq C, \tag{3.10}$$

then it will follow by (3.8) that

$$\sum_{i=0}^{T} N_i' \geq \frac{1}{2}\, N \left(e^{-\alpha t_T} - e^{-\alpha t_0}\right) \prod_{i=0}^{T} \left(1 - e^{-\gamma t_i}\right) > cN,$$

for some $c = c(\alpha, \nu)$.

3.2.2 The definition of T

Firstly, we will require that $t_i > B_1$ where $B_1 = B_1(\alpha, \nu, \varepsilon) > e$ is large enough so that we have

$$2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i \right) < (1-\alpha)\, t_i. \tag{3.11}$$

This condition implies that

$$\begin{aligned} t_i &\leq \lambda t_{i-1} + 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i \right) \overset{t_i \leq t_{i-1}}{<} \lambda t_{i-1} + 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_{i-1} \right) \\ &\overset{(3.11)}{<} (\lambda + 1 - \alpha)\, t_{i-1} = (2\alpha - 1 + 1 - \alpha)\, t_{i-1} = \alpha t_{i-1} < t_{i-1}. \end{aligned} \tag{3.12}$$

We use (3.12) in order to deduce that if $t_i > B_1$, then

$$e^{-\alpha(t_{i-1} - t_i)} < e^{-t_{i-1}\alpha(1-\alpha)}. \tag{3.13}$$

Given $\varepsilon > 0$, let $C = C(\alpha, \nu, \varepsilon) > B_1$ be such that

$$\begin{aligned}
&e^{-C\alpha(1-\alpha)} < \varepsilon, \\
&\frac{\nu(1-\varepsilon)^4}{4\pi} < C, \\
&\text{if } \lambda C < t - 2\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t\right) =: f(t), \text{ then } f(t) < t, \\
&\text{for all } t > C \text{ we have } 2\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t\right) < \frac{1-\alpha}{2} t, \\
&\frac{e^{-\gamma\lambda C/2}}{1 - e^{-\gamma(1-\alpha)C}} < \frac{1}{2},
\end{aligned} \tag{3.14}$$

where $\gamma = \gamma(\alpha, \varepsilon) > 0$ will be specified later. Let

$$T := \min\{i \ : \ t_i < C \text{ or } \Theta_i < \pi\}.$$

Thus, by (3.12) we deduce that

$$T = O(\log R). \tag{3.15}$$

Hence, (3.13) implies that for any $i < T$ we have

$$e^{-\alpha(t_{i-1} - t_i)} \overset{t_{i-1} \geq C}{<} \varepsilon. \tag{3.16}$$

The definition of T also implies that for $i < T$

$$t_i > \frac{\nu(1-\varepsilon)^4}{4\pi},$$

as required above. Recall that this ensures that the second term in the left-hand side of (3.5) is positive as

$$\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i\right) > 0,$$

and thereby (3.6) holds for all $i < T$.

Secondly, we will require that

$$\prod_{i=1}^{T}\left(1-e^{-\gamma t_i}\right) > \frac{1}{2}. \tag{3.17}$$

As we shall see in the next section, this will imply that

$$\Theta_i \geq \Theta_0 \prod_{j=1}^{T}\left(1-e^{-\gamma t_j}\right) > \pi. \tag{3.18}$$

As $\Theta_0 = 2\pi$ we need that the product on the right-hand side of the above is at least 1/2. This will be the case if

$$\sum_{j=1}^{T} e^{-\gamma t_j} < \frac{1}{2}.$$

To bound the above sum, we will give an upper bound on the difference of $t_j - t_{j-1}$. We have

$$t_j - t_{j-1} \overset{(3.12)}{<} (\alpha-1)t_{j-1} \overset{t_{j-1}>C, \alpha<1}{<} (\alpha-1)C.$$

Hence, we can write

$$\sum_{j=1}^{T} e^{-\gamma t_j} < e^{-\gamma t_T} \sum_{j=0}^{\infty} e^{-j\gamma(1-\alpha)C}.$$

Also by the third condition in (3.14)

$$\lambda C \leq \lambda t_{T-1} = t_T - 2\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_T\right) < t_T, \tag{3.19}$$

whereby

$$t_T > \lambda C. \tag{3.20}$$

Therefore,

$$\sum_{j=1}^{T} e^{-\gamma t_j} < \frac{e^{-\gamma\lambda C}}{1-e^{-\gamma(1-\alpha)C}} \overset{(3.14)}{<} \frac{1}{2}.$$

3.2.3 Some concentration results

In this sub-section, we will show that the number of vertices that belong to each band is almost determined. Note that by Lemma 2.1.5 (since $t_i \leq R/2$, for all $i \geq 0$) we have uniformly for all i

$$\mathbb{E}[N_i] = (1-o(1))N\alpha \int_{t_i}^{t_{i-1}} e^{-\alpha t} dt = (1-o(1))N\left(e^{-\alpha t_i} - e^{-\alpha t_{i-1}}\right). \tag{3.21}$$

(Here we take $t_{-1} = R$.) We need to show that this quantity grows fast enough as a function of N. To see that this is indeed the case, we write

$$\mathbb{E}[N_i] = (1-o(1))Ne^{-\alpha t_i}\left(1 - e^{-\alpha(t_{i-1}-t_i)}\right) \overset{(3.16)}{>} Ne^{-\alpha t_i}(1-\varepsilon). \tag{3.22}$$

Hence, since $t_i \leq R/2$, it follows that

$$\mathbb{E}[N_i] > (1-\varepsilon)\nu e^{\frac{1}{2}(1-\alpha)R} = \Omega(N^{1-\alpha}), \tag{3.23}$$

which tends to infinity as N grows, since $\alpha < 1$.

Hence, applying a standard Chernoff bound we deduce that with probability $1 - \exp\left(-\Omega\left(N^{1-\alpha}\right)\right)$ we have

$$N_i = (1 \pm \varepsilon)\mathbb{E}[N_i].$$

Hence, since $T = O(\ln R)$ (cf. (3.12)), a simple first-moment argument shows that with probability $1 - \exp\left(-\Omega\left(N^{1-\alpha}\right)\right)$ we have

$$N_i = (1 \pm \varepsilon)\mathbb{E}[N_i], \tag{3.24}$$

for all $0 \leq i \leq T$. In what follows, we shall condition on this event, which we denote by $\mathcal{N}$.

3.2.4 The inductive step

Throughout this section, we will have $\theta^{(i)} = 2(1-\varepsilon)e^{\frac{1}{2}(t_i+t_{i-1}-R)}$, for $0 < i \leq T$ and $\theta^{(0)} = 2(1-\varepsilon)$. Assume that there are N'_{i-1} vertices in the active area of $\mathcal{B}_{i-1}$. For a vertex in $v \in \mathcal{N}'_{i-1}$ let $S(v)$ denote the arc of angle $\theta^{(i)}$ around the projection of v on the circle of radius $R - t_i$ (in other words, the set of points of type t_i) - we denote this circle by $\mathcal{C}_i$. We call this the *shadow* of v. Let

$$S_i := \bigcup_{v \in \mathcal{N}'_{i-1}} S(v)$$

denote the union of the shadows of the vertices in $\mathcal{N}'_{i-1}$ — this is the active area of the band $\mathcal{B}_i$. Let Θ_i be the total angle of S_i.

We will determine Θ_i conditional on Θ_{i-1}, assuming that we have not specified which vertices among those in $\mathcal{N}_{i-1}$ belong to S_{i-1}. Let S'_{i-1} denote the projection of S_{i-1} on the circle $\mathcal{C}_i$. Note that S'_{i-1} is the disjoint union of arcs each of them having angle which is at least $\theta^{(i-1)}$. Moreover, the total angle covered by S'_{i-1} is Θ_{i-1} as well.

Assuming that the vertices of $\mathcal{N}_{i-1}$ have all type t_{i-1}, we expose their positions on $\mathcal{C}_{i-1}$ and consider the shadows of those points that will fall into S_{i-1}. Recall that this number is a stochastic lower bound on N'_{i-1}, whereby

$$\mathbb{E}\left[N'_{i-1} \mid N_{i-1}, \Theta_{i-1}\right] \geq N_{i-1}\frac{\Theta_{i-1}}{2\pi}.$$

Furthermore, since $\Theta_{i-1} \geq \pi$, as $i-1 < T$ and $N_{i-1} = \Omega(N^{1-\alpha})$, as the event $\mathcal{N}$ is realised, an application of the Chernoff bound implies that with probability $1 - o\left(\frac{1}{\ln N}\right)$

$$N'_{i-1} \geq N_{i-1}\frac{\Theta_{i-1}}{2\pi}(1-\varepsilon). \tag{3.25}$$

We will show that conditional on N'_{i-1} as above and Θ_{i-1} with high probability Θ_i is at least a certain fraction of Θ_{i-1}.

Lemma 3.2.2. *Conditional on N_{i-1} satisfying $\mathcal{N}$, on Θ_{i-1} as well as on N'_{i-1} satisfying (3.25), with probability $1 - o\left(\frac{1}{\ln N}\right)$ we have*

$$\Theta_i \geq \Theta_{i-1}\left(1 - e^{-\gamma t_i}\right),$$

for some constant $\gamma = \gamma(\alpha, \nu, \varepsilon) > 0$, uniformly for $i = 1, \ldots, T$.

Proof. To show this statement, we divide each subinterval of S'_{i-1} into segments of angle

$$\ell_{i-1} := \frac{\Theta_{i-1}}{N_{i-1}^2}.$$

It is possible that each of these subintervals contains at least one segment of smaller angle. However, each subinterval of S'_{i-1} contains many segments of angle ℓ_{i-1}. We denote by $\mathcal{P}$ the collection of all *those* segments. We will use a bounded-differences concentration inequality in order to show that with high probability most of them are contained in $S(v)$ for some $v \in \mathcal{N}'_{i-1}$.

Firstly, let us bound from below the size of $\mathcal{P}$. Recall that each subinterval of S'_{i-1} has angle at least $\theta^{(i-1)}$. Therefore, there are at most $\Theta_{i-1}/\theta^{(i-1)}$ subintervals. Each such subinterval contains at most one segment of angle less than ℓ_{i-1}. Hence,

$$|\mathcal{P}| \geq \left\lfloor \frac{\Theta_{i-1}}{\ell_{i-1}} \right\rfloor - \frac{\Theta_{i-1}}{\theta^{(i-1)}} \overset{\Theta_{i-1}<2\pi}{>} N_{i-1}^2 - \frac{2\pi}{\theta^{(i-1)}}. \tag{3.26}$$

But

$$\theta^{(i-1)} = 2(1-\varepsilon)e^{\frac{1}{2}(t_{i-1}+t_{i-2}-R)} \overset{t_{i-2}>t_{i-1}}{=\!\!>} 2(1-\varepsilon)e^{\frac{1}{2}(2t_{i-1}-R)} = 2\nu(1-\varepsilon)\frac{e^{t_{i-1}}}{N}, \tag{3.27}$$

and also since $\mathcal{N}$ is realised (3.22) implies that

$$N_{i-1} \geq (1-\varepsilon)^2 N e^{-\alpha t_{i-1}} = (1-\varepsilon)^2 \nu e^{R/2-\alpha t_{i-1}}.$$

Thus, in this case

$$N_{i-1}^2\theta^{(i-1)} \geq N_{i-1}\theta^{(i-1)} = 2\nu(1-\varepsilon)^3 e^{(1-\alpha)t_{i-1}} \overset{t_{i-1}\geq R/2}{\geq} 2\nu(1-\varepsilon)^3 e^{(1-\alpha)R/2}.$$

This together with (3.26) imply that

$$|\mathcal{P}| \geq N_{i-1}^2\left(1 - O\left(N^{\alpha-1}\right)\right). \tag{3.28}$$

It is now immediate that the total angle covered by the union of the segments in $\mathcal{P}$, which we denote by $\Theta_{\mathcal{P}}$, satisfies

$$\Theta_{\mathcal{P}} \geq \Theta_{i-1}\left(1 - O\left(N^{\alpha-1}\right)\right). \tag{3.29}$$

To be more precise, recall that each subinterval has angle that is at least $\theta^{(i-1)}$. Since the event $\mathcal{N}$ is realised, we have

$$\ell_{i-1} = \frac{\Theta_{i-1}}{N_{i-1}^2} \overset{(3.22)}{\leq} \frac{2\pi}{(1-\varepsilon)^2}\frac{e^{2\alpha t_i}}{N^2}.$$

Now, (3.27) implies that

$$\begin{aligned}\frac{\ell_{i-1}}{\theta^{(i)}} &< \frac{(1-\varepsilon)\pi}{\nu}\ \frac{e^{(2\alpha-1)t_i}}{N} \overset{1<2\alpha, t_i<t_0=R/2}{<} \frac{(1-\varepsilon)\pi}{\nu}\ \frac{e^{(2\alpha-1)R/2}}{N} \\ &= \frac{(1-\varepsilon)\pi}{\nu^2}\ e^{(\alpha-1)R} \overset{\alpha\leq 1}{=} o(1).\end{aligned}$$

In other words, uniformly for all $i < T$, we have $\theta^{(i-1)} > \theta^{(i)} \gg \ell_{i-1}$.

For a segment $\sigma \in \mathcal{P}$, let $\mathcal{E}_\sigma$ denote the event that the segment σ is not covered by $S(v)$, for all $v \in \mathcal{N}'_{i-1}$. The probability that the segment is indeed covered for a certain

$v \in \mathcal{N}'_{i-1}$ is at least $\frac{\theta^{(i)}}{\Theta_{i-1}} \geq \frac{\theta^{(i)}}{2\pi}$. Hence

$$\begin{aligned}\Pr\Big[\,\mathcal{E}_\sigma \mid N'_{i-1}, \Theta_{i-1}\,\Big] &\leq \left(1 - \frac{\theta^{(i)}}{2\pi}\right)^{N'_{i-1}} \leq \exp\left(-\frac{\theta^{(i)} N'_{i-1}}{2\pi}\right) \\ &\overset{(3.25)}{\leq} \exp\left(-(1-\varepsilon)\frac{\theta^{(i)}\Theta_{i-1}N_{i-1}}{4\pi^2}\right) \overset{(\Theta_{i-1}>\pi)}{\leq} \exp\left(-(1-\varepsilon)\frac{\theta^{(i)}N_{i-1}}{4\pi}\right).\end{aligned} \tag{3.30}$$

Now, on the event $\mathcal{N}$, the following holds through (3.22) and (3.24)

$$\theta^{(i)} N_{i-1} \geq 2\nu(1-\varepsilon)^3 e^{\frac{1}{2}(t_i + t_{i-1}) - \alpha t_{i-1}}.$$

But by (3.5) we have

$$\frac{1}{2}(t_i + t_{i-1}) - \alpha t_{i-1} = \ln\left(\frac{4\pi}{2\nu(1-\varepsilon)^4} t_i\right)$$

which, if substitute in (3.30) implies that

$$\Pr\Big[\,\mathcal{E}_\sigma \mid N'_{i-1}, \Theta_{i-1}\,\Big] \leq e^{-t_i}. \tag{3.31}$$

Let $\mathcal{P}'$ denote the subset of segments of $\mathcal{P}$ that are covered by $S(v)$, for some $v \in \mathcal{N}'_{i-1}$. Therefore,

$$\mu_{\mathcal{P}'} := \mathbb{E}\left[|\mathcal{P}'| \mid N'_{i-1}, \Theta_{i-1}\right] \geq |\mathcal{P}|\left(1 - e^{-t_i}\right). \tag{3.32}$$

Changing the position of one vertex in $\mathcal{N}'_{i-1}$ changes the number of these segments by at most $2\theta^{(i)}/\ell_{i-1}$. Hence, applying the Azuma-Hoeffdingf concentration bound (cf. [JŁR00] Theorem 2.25 p. 37) we deduce that

$$\Pr\Big[\,|\mathcal{P}'| < (1 - e^{-(1-\alpha)t_{i-1}/8})\mu_{\mathcal{P}'} \mid N'_{i-1}, \Theta_{i-1}\,\Big] = \exp\left(-\Omega\left(\frac{\mu^2_{\mathcal{P}'} e^{-(1-\alpha)t_{i-1}/4}\ell^2_{i-1}}{N'_{i-1}(\theta^{(i)})^2}\right)\right).$$

We will show now that

$$\frac{\mu^2_{\mathcal{P}'} e^{-(1-\alpha)t_{i-1}/4}\ell^2_{i-1}}{N'_{i-1}(\theta^{(i)})^2} = \Omega\left(N^{\frac{5}{4}(1-\alpha)}\right). \tag{3.33}$$

We will estimate the above quantity up to absolute multiplicative constants – we write $A \gtrsim B$ to denote that A/B is bounded from below by some constants that depend only on α, ν and ε. To derive the above lower bound we will need to deduce a stronger upper bound on t_i in terms of t_{i-1}. By (3.5) we have

$$\begin{aligned} t_i &= \lambda t_{i-1} + 2\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i\right) \overset{t_i < t_{i-1}}{\leq} \lambda t_{i-1} + 2\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_{i-1}\right) \\ &\overset{t_{i-1} > C,\ (3.14)}{<} \left(\lambda + \frac{1-\alpha}{2}\right) t_{i-1}. \end{aligned} \tag{3.34}$$

Now, we have

$$\begin{aligned} &\frac{\mu_{\mathcal{P}'}^2 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N'_{i-1}(\theta^{(i)})^2} \overset{(3.19),(3.22),(3.28)}{\gtrsim} \frac{N_{i-1}^4 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N_{i-1}(\theta^{(i)})^2} \\ &\gtrsim N_{i-1}^3 e^{-(1-\alpha)t_{i-1}/4} \left(\frac{\Theta_{i-1}}{N_{i-1}^2}\right)^2 \frac{N^2}{e^{t_i + t_{i-1}}} \overset{(3.17)}{\gtrsim} \frac{N^2}{N_{i-1}} e^{-(5-\alpha)t_{i-1}/4 - t_i} \\ &\overset{(3.22),\ t_i \leq t_{i-1}}{\gtrsim} N e^{-(5-\alpha)t_{i-1}/4 - t_i + \alpha t_{i-1}} \overset{(3.34)}{\gtrsim} N e^{(-(5-\alpha)/4 - \lambda + (1-\alpha)/2 + \alpha) t_{i-1}}. \end{aligned}$$

But

$$-(5-\alpha)/4 - \lambda + (1-\alpha)/2 + \alpha = -(5-\alpha)/4 - 2\alpha + 1 + (1-\alpha)/2 + \alpha = 1/4 - 5\alpha/4.$$

Hence,

$$\frac{\mu_{\mathcal{P}'}^2 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N'_{i-1}(\theta^{(i)})^2} \gtrsim N e^{(1-5\alpha)t_{i-1}/4} \overset{t_{i-1} \leq R/2, \alpha > 1/2}{\gtrsim} e^{\frac{5}{4}(1-\alpha)\frac{R}{2}} \gtrsim N^{\frac{5}{4}(1-\alpha)}. \tag{3.35}$$

Now,

$$\Theta_i \geq \ell_{i-1} |\mathcal{P}'|.$$

Hence, conditional on N'_{i-1} and Θ_{i-1} with probability $1 - o\left(\frac{1}{\ln N}\right)$ we have

$$\Theta_i \geq \ell_{i-1} \mu_{\mathcal{P}'} (1 - e^{-\left(\alpha - \frac{1}{2}\right) t_{i-1}/2}).$$

We will bound the right-hand side of the above from below as follows:

$$\begin{aligned}
&\ell_{i-1}\mu_{\mathcal{P}'}(1-e^{-\left(\alpha-\frac{1}{2}\right)t_{i-1}/2}) \overset{(3.32)}{\geq} \ell_{i-1}|\mathcal{P}|(1-e^{-t_i})(1-e^{-\left(\alpha-\frac{1}{2}\right)t_{i-1}/2}) \\
&\overset{(3.28)}{\geq} \ell_{i-1}N_{i-1}^2\left(1-O\left(N^{\alpha-1}\right)\right)(1-e^{-t_i})(1-e^{-\left(\alpha-\frac{1}{2}\right)t_{i-1}/2}) \\
&= \Theta_{i-1}\left(1-O\left(N^{\alpha-1}\right)\right)(1-e^{-t_i})(1-e^{-\left(\alpha-\frac{1}{2}\right)t_{i-1}/2}) \\
&\geq \Theta_{i-1}\left(1-O\left(N^{\alpha-1}\right)\right)\left(1-e^{-t_i}-e^{-\left(\alpha-\frac{1}{2}\right)t_{i-1}/2}\right).
\end{aligned} \tag{3.36}$$

But by (3.5) we have

$$\frac{1}{2}\left(\alpha-\frac{1}{2}\right)t_{i-1} = \frac{1}{4}t_i - \frac{1}{2}\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4}t_i\right) \overset{(3.14),(3.20)}{>} \frac{1}{4}(1-\varepsilon)t_i.$$

We substitute this bound into the last expression of (3.36) and deduce the following: there exists a constant $\gamma = \gamma(\varepsilon) > 0$ such that for all N sufficiently large and for all $i = 1, \dots, T$ we have

$$\Theta_i \geq \Theta_{i-1}\left(1-e^{-\gamma t_i}\right).$$

□

3.2.5 Proof of Theorem 1.6.1

For $i = 1, \dots, T$ let $\mathcal{E}_i$ denote the event that for all $0 \leq j \leq i$, we have

$$\Theta_j \geq \Theta_{j-1}(1-e^{-\gamma t_j})$$

and

$$N'_j \geq N_j\frac{\Theta_j}{2\pi}(1-\varepsilon).$$

Note that conditional on $\mathcal{N}$, the latter inequality together with (3.21) and (3.24) implies that for any N sufficiently large

$$N'_j \geq (1-\varepsilon)^3 N\left(e^{-\alpha t_j}-e^{-\alpha t_{j-1}}\right)\frac{\Theta_j}{2\pi}.$$

Now by (3.25) and Lemma 3.2.2, we have

$$\Pr\Big[\,\mathcal{E}_i \mid \mathcal{E}_{i-1}, \mathcal{N}\,\Big] = 1 - o\left(\frac{1}{R}\right).$$

But as the sequence $\{t_i\}_{i=1,\ldots,T}$ decreases exponentially fast (cf. (3.12)), we have $T = O(\ln R)$. Hence, since the events $\{\mathcal{E}_i\}_{i=1,\ldots,T}$ form a decreasing sequence, we deduce that

$$\Pr\Big[\,\mathcal{E}_T \mid \mathcal{N}\,\Big] = 1 - o(1).$$

On the event $\mathcal{E}_T$, we have $\Theta_i > \pi$ for all $i = 1, \ldots, T$ (cf. (3.18)). Therefore,

$$N_i' \geq \frac{1}{2}\,(1-\varepsilon)^3 N \left(e^{-\alpha t_i} - e^{-\alpha t_{i-1}}\right).$$

which in turn implies that

$$\sum_{i=1}^{T} N_i' \geq \frac{1}{2}\,(1-\varepsilon)^3 N \left(e^{-\alpha t_T} - e^{-\alpha t_0}\right) > \frac{1}{2}\,(1-\varepsilon)^3 N \left(e^{-\alpha\lambda C/2} - o(1)\right).$$

3.3 Proof of Theorem 1.6.2

In the critical case, that is, when $\alpha = 1$, the probability of having a giant component turns out to depend on the value of ν. It will be convenient to work with the *Poisson model* $\mathcal{P}(N; \alpha, \nu)$ as defined in Section 2.2.

3.3.1 The subcritical case

We prove the first part of Theorem 1.6.2 for $\mathcal{P}(N; \alpha, \nu)$ by contradiction. Assuming we have a component of size $\frac{N}{\log\log R}$, then at least $\frac{N}{2\log\log R}$ vertices of type at most $T = \log\log R$ must be contained in that component as a.a.s. at most $\frac{N}{2\log\log R}$ vertices have type larger than T. We will use a *smooth breadth exploration process* starting at a point v_0 of type at most $\log\log R$, a continuous-type version of the process defined in Chapter 2.3 as opposed to the discretised version used in Section 3.1:

- Let $T_0^a = T_0^c = t_{v_0}$ and $v_0^a = v_0^c = v_0$.

- Do the following three steps recursively for $i \geq 1$.

- Let N_i^a be the set of neighbours of v_i^a or v_i^c that are in anticlockwise direction of v_i^a, and let N_i^c be the set of neighbours of v_i^a or v_i^c that are in clockwise direction of v_i^c.

- If N_{i-1}^c is empty, set $v_i^c = v_{i-1}^c$. Otherwise, let v^c be the vertex of highest type in N_{i-1}^c and let T_i^c be the type of v^c. Let θ_i^c be the maximum relative angle between v_{i-1}^c and any adjacent point of type T_i^c. Define v_i^c as the point of type T_i^c with relative angle θ_i^c from v_{i-1}^c in the clockwise direction. We call this the ith clockwise root vertex.

- Analogue to the previous step but using the anticlockwise direction and the vertex v_{i-1}^a, define the ith anticlockwise root vertex v_i^a.

- There are three stopping conditions for the process:

  (i) N_i^a and N_i^c are empty;

  (ii) $T_i^a > \log R$ or $T_i^c > \log R$;

  (iii) $i = \log^2 \log R$ or the angle between v_i^a and v_0 or v_i^c and v_0 exceeds π.

We define stopping times τ_1 and τ_2 as the stopping times that correspond to the first and the second stopping conditions, respectively. Because of the third stopping condition, we know that $0 \leq \tau_i \leq \log\log R$ or $\tau_i = \infty$ for $i = 1, 2$.

The following two lemmas show that the process stops quickly for a suitable choice of ν.

Lemma 3.3.1. *For $\nu < \frac{\pi}{8}$, starting at a vertex of type at most $\log\log R$, we have*

$$\mathbb{P}(\tau_1 < \infty) = 1 - o\left(\left(\frac{\log\log R}{\log R}\right)^c\right) \text{ for some } c > 0.$$

Lemma 3.3.2. *For $\nu < \frac{\pi}{8}$, starting at a vertex of type at most* $\log\log R$*, we have*

$$\mathbb{P}(\tau_2 < \infty) = o\left(\frac{\log^2 \log R}{R^c}\right) \text{ for some } 0 < c < 1.$$

Parametrising with respect to the initial vertex v, we denote the above stopping times by $\tau_1(v)$ and $\tau_2(v)$. Let also $\tau(v)$ denote the stopping time of the process.

Using the two lemmas we can prove the first part of Theorem 1.6.2.

Proof of the first part of Theorem 1.6.2. In what follows, we assume that N is large enough for all our estimates to hold. For simplicity, we shall denote the set of vertices by V_N - note that this is now a random set of vertices as described above. Let $V_{high} = \{v \in \mathrm{V}_N \ : \ t_v \geq \log\log R\}$ and $V_{low} = \mathrm{V}_N \setminus V_{high}$. In other words, we partition the set of vertices into two parts: that of vertices of low type and that of vertices of high type. Now, we set

$$V_{small} = \{v \in V_{low} \ : \ \tau_1(v) < \infty, \tau_2(v) = \infty\},$$

and $V_{large} = V_{low} \setminus V_{small}$. The smooth breadth exploration process started at a vertex $v \in V_{small}$ terminates after at most $\log^2 \log R$ steps and has only had root vertices of type at most $\log R$. In this case, by Lemma 2.1.7 the angle gained at every single step is at most $2.5 \exp(\log R - R/2)$ in either direction. Hence, the total angle gained during the process in both directions is at most $5 \exp(\log R - R/2) \log^2 \log R$. (This justifies the name V_{small}: the total angle of the component of v is bounded by this expression which is a decaying function of N.)

Now, Lemmas 3.3.1 and 3.3.2 imply that there exists a positive constant c such that for a vertex $v \in V_{low}$ we have $\mathbb{P}(v \in V_{large} \mid v \in V_{low}) = o\left(\left(\frac{\log\log R}{\log R}\right)^c\right)$. Thereby,

$$\mathbb{E}(|V_{large}|) = o\left(N \left(\frac{\log\log R}{\log R}\right)^c\right).$$

Hence, by Markov's inequality, a.a.s.

$$|V_{large}| \le \frac{N}{2\log\log R}.$$

Also, using Lemma 2.1.5 and the concentration of the Poisson distribution, we can deduce that a.a.s.

$$|V_{high}| \le \frac{N}{2\log\log R}.$$

Thus, a.a.s.

$$|V_{large} \cup V_{high}| \le \frac{N}{\log\log R}.$$

Now, by Lemma 3.1.7, a.a.s. every vertex in V_{small} is contained in a component with at most $10N\exp(\log R - R/2)\log^2\log R < 10\nu R\log^2\log R$ vertices. Hence, any component of large size must be induced by vertices in $V_{large} \cup V_{high}$, whereby $|L_1| \le \frac{N}{\log\log R}$. □

To prove the lemmas, we simplify the process in a way that allows for any real types, dropping the $0 \le t \le R$ requirement. Let $T_i = \max\{T_i^a, T_i^c\}$. We use the following cdf and pdf for T_i given the maximum T_{i-1} of the types of v_{i-1}^a and v_{i-1}^c:

$$F_{T_i}(t) = \exp\left(-4\nu\frac{1.01}{\pi}e^{\frac{1}{2}(t_{i-1}-t)}\right) \tag{3.37}$$

$$f_{T_i}(t) = 2\nu\frac{1.01}{\pi}e^{\frac{1}{2}(t_{i-1}-t)}\exp\left(-4\nu\frac{1.01}{\pi}e^{\frac{1}{2}(t_{i-1}-t)}\right). \tag{3.38}$$

Claim 3.3.3. *Lemmas 3.3.1 and 3.3.2 hold if they hold for the extended and simplified distribution of types in Equation (3.37), where not finding a next neighbour in the original distribution corresponds to a negative type in the extended one.*

Proof. We prove the result by showing that the given cdf is a lower bound on the actual cdf at any point. This means there is a coupling in which any vertex of the actual distribution is coupled with a vertex of higher or equal type and same angle in the simplified distribution. We later prove that the distance we change the type by does not depend on the type of the active vertex for the vertices that we consider, so a higher type cannot have a negative influence on the result we want to prove.

Note that the number of vertices in $N_i^a \cup N_i^c$ is stochastically dominated by the number of neighbours of a vertex of type T_i (when no part of the relevant area has been exposed). This is simply because the vertices in N_i^a must necessarily all be in the anticlockwise side of the outer tube around the vertex of type T_i that is of relative angle 0 with v_i^a and the vertices in N_i^c must all be in the clockwise side of the outer tube around the vertex of type T_i that is of relative angle 0 with v_i^c. These two sides together have precisely the same distribution of vertices in them as a tube around a vertex of type T_i. From now on we thus consider just this distribution.

If the type of a vertex is less than $\log R$, we can use Lemma 2.1.7 with $\varepsilon = 0.009$ to get a bound on the relative angle for possibly adjacent vertices up to type $R - 2\log R < R - \log R - c_0(0.009)$ for sufficiently large N. Note that the expected number of vertices of type larger than $R - 2\log R$ is

$$N \frac{\cosh(2\log R) - 1}{\cosh(R) - 1} = o(1).$$

Thus the probability of having such a vertex is $o(1)$ and we can condition on no such vertex existing. We use outer tubes to estimate the expected value of $N_{t_0}^t$, the number of neighbours of type at least t of a vertex of type $t_0 \leq \log R$, using the outer tubes for $\varepsilon = 0.009$ as an upper bound, taking N large enough and using Lemma 2.1.5:

$$\begin{aligned}
\mathbb{E}N_{t_0}^t &\lesssim N \int_t^{R-2\log R} 2\frac{1.009}{\pi} e^{\frac{1}{2}(t_0+t'-R)} e^{-t'} dt' \\
&\leq \nu e^{\frac{1}{2}R} \int_t^{R-2\log R} 2\frac{1.01}{\pi} e^{\frac{1}{2}(t_0+t'-R)} e^{-t'} dt' \\
&= 2\nu \frac{1.01}{\pi} e^{\frac{t_0}{2}} \int_t^{R-2\log R} e^{-\frac{1}{2}t'} dt' \\
&\leq 2\nu \frac{1.01}{\pi} e^{\frac{t_0}{2}} \int_t^{\infty} e^{-\frac{1}{2}t'} dt' \\
&= 4\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_0-t)}.
\end{aligned}$$

With this, as we are using the Poisson distribution, we get the following cdf for the distribution of the next type, given we are at step i with type t_0:

$$\begin{aligned}F_{T_i}(t) &= \mathbb{P}(T_i \leq t | T_{i-1} = t_0) = \mathbb{P}(|N_{t_0}^t| = 0)\\&= \exp(-\mathbb{E}(N_{t_0}^t)) \geq \exp\left(-4\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_0 - t)}\right).\end{aligned}$$

□

To prove the two lemmas we need to introduce new notation. Instead of looking at the types of the vertices at some step, we analyse the jump $J_i = T_i - T_{i-1}$ in some step, the difference in types from one step to the next. This makes sense as $F_{T_i}(t)$ only depends on the difference of T_i and t, each jump is distributed as

$$\begin{aligned}F_J(j) &= \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{j}{2}}\right)\\f_J(j) &= 2\nu \frac{1.01}{\pi} e^{-\frac{j}{2}} \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{j}{2}}\right).\end{aligned}$$

Starting at a vertex of type $T_0 (= \log \log R)$, we write

$$T_i = T_0 + \sum_{k=1}^{i} J_k,$$

where we couple with a sequence of independent random variables having as their cdf the function F_J. The type T_i is thus coupled with the sum of independent copies of the jump.

We are now ready to prove Lemma 3.3.2.

Proof of Lemma 3.3.2. We first calculate the following expectation: for $s > 0$ we have

$$\begin{aligned}\mathbb{E}e^{sJ_k} &= \int_{-\infty}^{\infty} e^{sx} 2\nu \frac{1.01}{\pi} e^{-\frac{x}{2}} \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{x}{2}}\right) dx\\&= 2\nu \frac{1.01}{\pi} \int_{-\infty}^{\infty} e^{(s-\frac{1}{2})x} \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{x}{2}}\right) dx.\end{aligned}$$

Changing variables $y = 4\nu\frac{1.01}{\pi}e^{-\frac{x}{2}}$, $dy = -2\nu\frac{1.01}{\pi}e^{-\frac{x}{2}}dx$ we get $e^{sx} = \left(4\nu\frac{1.01}{\pi y}\right)^{2s}$ and

$$\begin{aligned}\mathbb{E}e^{sJ_k} &= -\int_{\infty}^{0}\left(4\nu\frac{1.01}{\pi y}\right)^{2s} e^{-y}dy = \left(4\nu\frac{1.01}{\pi}\right)^{2s}\int_0^{\infty} y^{-2s}e^{-y}dy \\ &= \left(4\nu\frac{1.01}{\pi}\right)^{2s}\int_0^{\infty} y^{(1-2s)-1}e^{-y}dy = \left(4\nu\frac{1.01}{\pi}\right)^{2s}\Gamma(1-2s).\end{aligned}$$

Given a starting vertex of type $\log\log R$, we calculate, for large N, the probability of reaching type at least $\log R$ in any step $i < \log\log R$. For $s > 0$ arbitrary we have

$$\begin{aligned}\mathbb{P}(T_i \geq \log R) &= \mathbb{P}(T_0 + \sum_{k=1}^{i} J_k \geq \log R) \leq \mathbb{P}(\sum_{k=1}^{i} J_k \geq \frac{1}{2}\log R) \\ &= \mathbb{P}(e^{s\sum_{k=1}^{i} J_k} \geq e^{\frac{s}{2}\log R}) \\ &\leq \mathbb{E}e^{s\sum_{k=1}^{i} J_k}e^{-\frac{s}{2}\log R} \\ &= e^{-\frac{s}{2}\log R}\prod_{k=1}^{i}\mathbb{E}e^{sJ_k},\end{aligned}$$

using Markov's inequality. Choosing $0 < s < \frac{1}{2}$ arbitrarily, we get some constant $C > 1$ such that $\mathbb{E}e^{sJ_k} = C$. Thus

$$\mathbb{P}(T_i \geq \log R) \leq e^{-\frac{s}{2}\log R}C^i \leq e^{-\frac{s}{2}\log R}C^{\log^2\log R} = o(R^{-c}),$$

for some $0 < c = c(s) < 1$. With this, we can use the union bound to bound the probability that the smooth breadth exploration process has type at least $\log R$ at one of the first $\log^2\log R$ steps:

$$\mathbb{P}(\exists 0 < i \leq \log^2\log R : T_i \geq \log R) \leq \sum_{i=1}^{\log^2\log R}\mathbb{P}(T_i \geq \log R) = o\left(\frac{\log^2\log R}{R^c}\right).$$

$\square$

We use the same technique to prove Lemma 3.3.1.

Proof of Lemma 3.3.1. We would like to determine the probability that the process

$(T_i)_{i\geq 0}$ crosses 0 by step $M := \log^2 \log R$. We have

$$\begin{aligned}\mathbb{P}(T_M \geq 0) &= \mathbb{P}(T_0 + \sum_{k=1}^{M} J_k \geq 0) \leq \mathbb{P}(\sum_{k=1}^{M} J_k \geq -\log\log R) \\ &\overset{s\geq 0}{=} \mathbb{P}(e^{s\sum_{k=1}^{M} J_k} \geq e^{-s\log\log R}) \\ &\leq (\log R)^s \prod_{k=1}^{M} \mathbb{E}e^{sJ_k},\end{aligned}$$

again using Markov's inequality. As $\nu < \frac{\pi}{8}$ and using the calculations from the proof of Lemma 3.3.2, we can find $0 < s < \frac{1}{2}$ and $0 < \hat{c} < 1$ such that for $1 \leq k \leq M$

$$\mathbb{E}e^{sJ_k} \leq \left(4\nu\frac{1.01}{\pi}\right)^{2s} \Gamma(1-2s) = \hat{c} < 1.$$

With this we have

$$\mathbb{P}(T_M \geq 0) \leq (\log R)^s \hat{c}^{\log^2 \log R} = o\left(\left(\frac{\log\log R}{\log R}\right)^c\right),$$

for some $c > 0$. Thus $\mathbb{P}(T_M < 0) = 1 - o\left(\left(\frac{\log\log R}{\log R}\right)^c\right)$. □

3.3.2 The supercritical case

For the second part of Theorem 1.6.2, we split the disk $\mathcal{D}_R$ into cells so that the expected number of vertices in each cell is constant. Furthermore, the cells are defined so that if two neighbouring cells contain at least one vertex each, then these vertices are adjacent. With some rules for adjacencies of the cells we can then explore possible components and estimate the number of vertices in them by the number of cells that correspond to them.

For the discretisation, we define the sequence $t_i = i\log 2$ for $0 \leq i \leq \left\lceil \frac{R}{4\log 2} \right\rceil =: T$. Using this, we define the bands $B_i = \{p \in \mathcal{D}_R : t_{i-1} \leq t_p < t_i\}$, for $1 \leq i \leq T$. Let $M = 2^T \left\lceil \frac{2\pi}{2^T 0.95 e^{-R/2}} \right\rceil$. For each i, we split band B_i into $M/2^{i-1}$ cells $C_j^{(i)}$, starting at angle 0 with $j = 1$. Note that the definition of M implies that the projection of a cell in

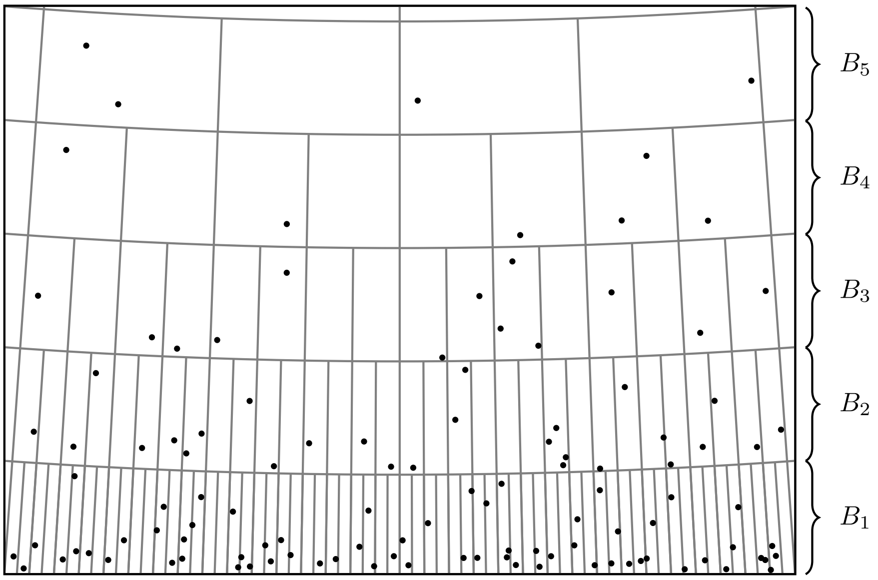

Figure 3.2: The discretisation of $\mathcal{D}_R$.

B_i is exactly split into two in B_{i-1}. See Figure 3.2 for an example of the discretisation. As $2^T = o(e^{\frac{R}{2}})$, it follows that $M = 2^T \frac{2\pi}{2^T 0.95 e^{-R/2}}(1+o(1))$, whereby the number of cells in band i is $(1+o(1))\frac{2\pi e^{R/2}}{2^{i-1}0.95}$. We claim the following property:

Lemma 3.3.4. *Any vertex in cell $C_j^{(i)}$ is adjacent to any vertex in any of the cells $C_{j-1}^{(i)}$, $C_{j+1}^{(i)}$, $C_{2\cdot j-1}^{(i-1)}$, $C_{2\cdot j}^{(i-1)}$ and $C_{\lceil j/2\rceil}^{(i+1)}$, where we let $C_0^{(i)} = C_{M/2^{i-1}}^{(i)}$ and $C_{M/2^{i-1}+1}^{(i)} = C_1^{(i)}$.*

Proof. This configuration of cells is illustrated on the left part of Figure 3.3. As we consider vertices of type at most $R/4 + \log 2$, Lemma 2.1.7 implies that any two of these vertices are connected if their relative angle is at most $1.92e^{1/2(t_u+t_v-R)}$, provided that N is large enough.

The vertices in $C_j^{(i)}$ have type at least $t_{i-1} = (i-1)\log 2$ and the relative angle of a vertex v in $C_{j-1}^{(i)}$ and u in $C_j^{(i)}$ is at most $2^{i-1}\ 2\cdot 0.96\ e^{-R/2} = 1.92e^{-R/2+(i-1)\log 2} \leq 1.92e^{1/2(t_u+t_v-R)}$, so v and u are adjacent. The same bounds on the maximum relative angle and minimum type hold for vertices u' in $C_j^{(i)}$ and v' in $C_{\lceil j/2\rceil}^{(i+1)}$, so u' and v' are adjacent. By simple change of variables we get all the desired adjacencies. $\square$

We define two auxiliary graphs, a blue graph G_b and a red graph G_r. The vertices of the blue graph will be those cells that contain at least one vertex, whereas the vertices

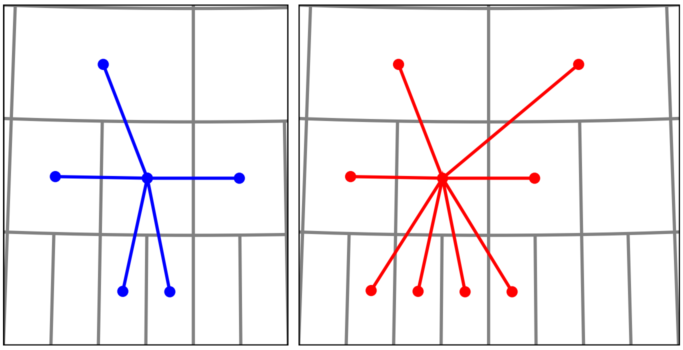

Figure 3.3: Blue(left) and red(right) neighbourhood.

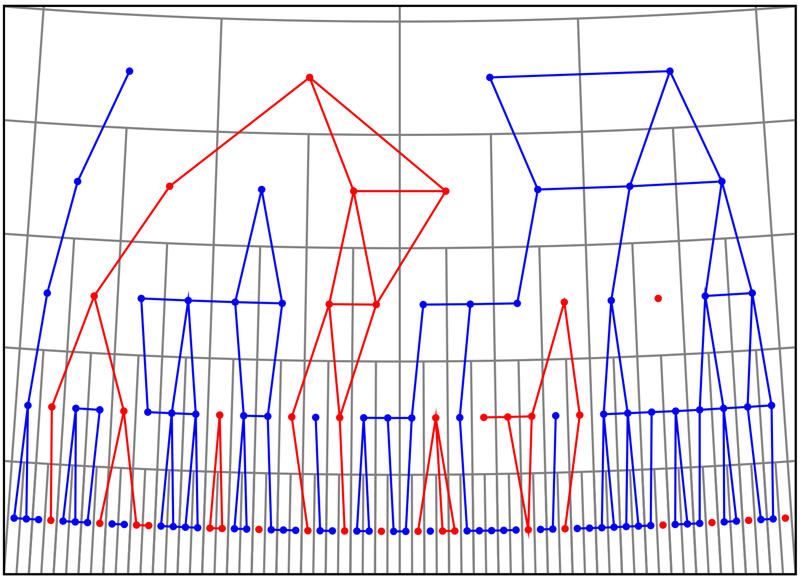

Figure 3.4: G_c resulting from the Graph in Figure 3.2.

of the red graph are those that contain no vertex. Two vertices of G_b are connected if the corresponding cells share an edge. This means that vertices of $\mathcal{P}(N;\alpha,\nu)$ in the cells corresponding to adjacent vertices in G_b are themselves adjacent by Lemma 3.3.4. Two vertices of G_r are connected if their corresponding cells share at least a point. This corresponds to the same adjacency as in G_b but with added "diagonal" edges. These adjacencies are illustrated in Figure 3.3. We denote by G_c the union of the two graphs. For the graph resulting from the example in Figure 3.2, see Figure 3.4. Whenever necessary we will refer to the vertices of the graphs as cells. A *blue component* is a component of G_c that consists of blue vertices. A *red path* is a path in G_c that consists of red vertices. These are the main structures we use, but we also use the red/blue notation for other structures. Because of the adjacency rules, a blue component is always surrounded by a red path or a collection of red paths, the periphery and the inside of the disk. We are now interested in the probability of a vertex being blue or red. Note that

$$M = 2^T \left\lceil \frac{2\pi}{2^T 0.95 e^{-R/2}} \right\rceil \leq \frac{2\pi}{0.95e^{-R/2}} + 2^T \leq \frac{2\pi + 2^T e^{-R/2}}{0.95e^{-R/2}}. \tag{3.39}$$

Claim 3.3.5. *The probability of a vertex in G_c being red is at most $e^{-\frac{\nu}{5\pi}}$, the probability of being blue is at least $1 - e^{-\frac{\nu}{5\pi}}$.*

Proof. Let $v_{i,j}$ be a vertex of G_c that corresponds to $C_i^{(j)}$. Let $P_i^{(j)}$ be the number of vertices in $C_i^{(j)}$. Using Lemma 2.1.5 and the above upper bound on M (3.39), we have for large N

$$\begin{aligned}
\mathbb{E}(P_i^{(j)}) &= N \int_{t_{i-1}}^{t_i} \frac{2^{i-1}}{M}(1+o(1))e^{-t}dt \\
&\geq 0.94\nu e^{R/2} \frac{2^{i-1}e^{-R/2}}{2\pi + 2^T e^{-R/2}} \int_{t_{i-1}}^{t_i} e^{-t}dt \\
&\geq 0.94\nu \frac{2^{i-1}}{2\pi + 2^{\frac{R}{4\log 2}+1}e^{-R/2}}(e^{-t_{i-1}} - e^{-t_i}) \\
&= 0.94\nu \frac{2^{i-1}}{2\pi + 2e^{-R/4}}(e^{-(i-1)\log 2} - e^{-i\log 2}) \\
&= 0.94\nu \frac{2^{i-1}}{2\pi + 2e^{-R/4}} 2^{-i}(2-1) \\
&\geq \frac{\nu}{5\pi},
\end{aligned}$$

for N large enough. Since the number of vertices in $C_i^{(j)}$ follows a Poisson distribution with parameter $\mathbb{E}(P_i^{(j)})$, we have

$$\mathbb{P}(v_{i,j} \text{ red}) = \mathbb{P}(P_i^{(j)} = 0) = e^{-\mathbb{E}(P_i^{(j)})} \leq e^{-\frac{\nu}{5\pi}}$$

$$\mathbb{P}(v_{i,j} \text{ blue}) = \mathbb{P}(P_i^{(j)} > 0) = 1 - e^{-\mathbb{E}(P_i^{(j)})} \geq 1 - e^{-\frac{\nu}{5\pi}}$$

□

Our aim is now to show that there is a blue component of εN cells for some $\varepsilon > 0$. Note that this implies that a giant component in $\mathcal{P}(N;\alpha,\nu)$ exists as each of the εN cells contain at least one vertex and the adjacency rules of the vertices of G_c imply that these vertices induce a connected component in $\mathcal{P}(N;\alpha,\nu)$. Or main tool for proving this is the following lemma that bounds the maximum length of a red path, using an argument similar to Peierls' argument from percolation theory (see [Gri99] or [Pen03]).

Lemma 3.3.6. *Let $\nu > 20\pi$. A.a.s. all red paths have length at most $L := T - 1 = \left\lceil \frac{R}{4\log 2} \right\rceil - 1$.*

Proof. Note that any cell in the red graph has at most 8 neighbours. The number of cells is at most $M(1+1/2+1/4+\cdots) = 2M \overset{(3.39)}{<} \frac{4\pi}{0.94}e^{R/2} < 5\pi e^{R/2}$. As $\nu > 20\pi$, this implies that the number of cells is at most N. Let $p = \mathbb{P}(v_{i,j} \text{ blue}) \geq 1 - e^{-\frac{\nu}{5\pi}} \geq 1 - e^{-\frac{20\pi}{5\pi}} > 0.98$ by Claim 3.3.5. Thus $8(1-p) < 0.2$. Let $P_c(\ell)$ be the number of red paths starting at cell c and having length ℓ. We have

$$\mathbb{E}(P_c(\ell)) \leq 8^\ell (1-p)^\ell = (8(1-p))^\ell < 0.2^\ell.$$

Since the number of cells is at most N we thus can bound the expected number of red paths of length ℓ by $0.2^\ell N$. Let c be such that $1 + c = \frac{\log 5}{2\log 2} \simeq 1.16$. The expected number of paths of length at least L is

$$\begin{aligned}
N\sum_{\ell \geq L} 0.2^\ell &= O\left(N\ 0.2^L\right) = O\left(e^{R/2} 0.2^{\frac{R}{4\log 2}}\right) \\
&= O\left(e^{R/2} e^{-\log 5\left(\frac{R}{4\log 2}\right)}\right) \\
&= O\left(e^{R/2} e^{-(1+c)\frac{R}{2}}\right) = O\left(\nu e^{-c\frac{R}{2}}\right) = o(1).
\end{aligned}$$

This means the probability of having a red path of length at least L is $o(1)$. □

With this we are able to make statements on the structure of G_c. We prove that a.a.s. there is a *blue lollipop* L_b: a blue cycle surrounding the origin of $\mathcal{D}_R$ that contains

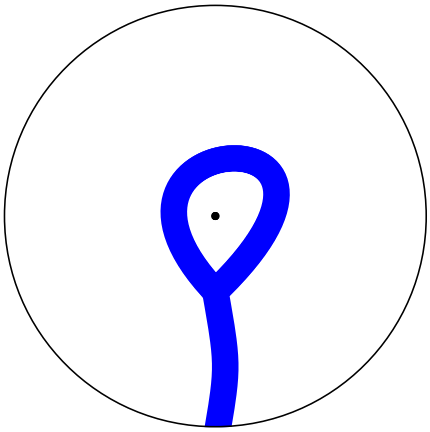

Figure 3.5: Blue lollipop.

a cell in B_1 or is connected to such a cell by a blue path (see Figure 3.5). We call the relevant cell of B_1 the *base* of the lollipop.

Claim 3.3.7. *Let $\nu > 20\pi$. A.a.s. G_c contains a blue lollipop.*

Proof. By Lemma 3.3.6, a.a.s. there is no red path of length L. The number of bands is $T > L$, so a.a.s. there is no red path connecting B_1 and B_T. This implies that there must be a blue cycle C surrounding the origin of $\mathcal{D}_R$. Now, the number of cells in any band is at least

$$M/2^T \geq \frac{2\pi}{2^T e^{-R/2}} \geq \frac{2\pi}{2^{\frac{R}{4\log 2}+1} e^{-R/2}} = \pi e^{R/4} \geq L$$

for N sufficiently large, as $L = O(R)$. Thus any red cycle surrounding C would have length at least L. But by Lemma 3.3.6 a.a.s. there is no such cycle. This implies that C must either contain a cell of B_1 or there must be a blue path P connecting C and some cell in B_1. In either case we have a blue lollipop. □

Starting at the base c_1 of the lollipop we will consider the following process traversing the band B_1 in clockwise direction:

- Starting with $i = 1$, do the following steps until c_1 is reached again.
- Let c_i' be the first red cell in B_1 in clockwise direction from c_i. Let S_i be the number of blue cells from c_i to c_i', including c_i.

- Let r_i be the red cell on B_1 that is farthest away from c_i' in clockwise direction and that is connected to c_i' via a red path - if such a cell does not exist, then $r_i = c_i'$. Denote by R_i the number of cells of B_1 between c_i' and r_i in clockwise direction, including these two. Let c_{i+1} be the cell succeeding r_i.

The process will end at some index $i = K$. Assume that S_K is the number of blue cells between c_K and c_1 in clockwise direction, including c_K if $c_K \neq c_1$ but excluding c_1. Note that the number of cells in B_1 is $M = 2^T \left\lceil \frac{2\pi}{2^T 0.95 e^{-R/2}} \right\rceil \geq \frac{2\pi}{\nu} N$. This means we have $S_K + \sum_{i=1}^{K-1}(S_i + R_i) = M \geq \frac{2\pi}{\nu} N$. We will prove that a.a.s. K is linear in N. We begin with the following properties:

Claim 3.3.8.

1. *Any two cells c_i and c_j are connected by a blue path for $1 \leq i, j \leq K$.*

2. *If the red path connecting c_i' and r_i has length ℓ, then $R_i < 2^{\ell/2+2}$.*

Proof.

1. Let $\hat{c}$ be the blue cell preceding c_1'. If there was no blue path connecting $\hat{c}$ to c_2 then there would be a red path originating at a red cell between c_1' and c_2 and ending in a red cell $\hat{r}_1$ that is farther in the anticlockwise direction from c_1' than c_2, thus also farther than r_1. But this means this red path must meet the path from c_1' to r_1, creating a path from c_1' to $\hat{r}_1$ — a contradiction. This means that $\hat{c}$ is connected to c_2 via a blue path. But all the cells between c_1 and $\hat{c}$ are blue, so there is a blue path from c_1 to c_2. Similarly, we can show that c_i is connected to c_{i+1}.

2. We fix ℓ and want to find the path of length ℓ between c_i' and r_i that “encloses" most cells in B_1. Note that the cells in a band become half as many when we increase the index of the band by one. We claim that the optimal path for this choice increases the band by one with each of its first $\lfloor \ell/2 \rfloor$ edges, stays in the same band for one edge if ℓ is odd, and then decreases the band by one for the

remaining $\lfloor \ell/2 \rfloor$ edges, such as the one in Figure 3.6. Assume this is not the case, then at least one of the following must be true:

- There is an edge between two cells at the same band that is not of the highest index among all cells in the path. Then, taking away this edge and instead inserting one at the highest index, thus shifting the remaining path, means that at least one more cell is covered with a path of the same length, a contradiction.
- There is an index i such that there are two cells c_1 and c_2, both in the band B_i, that are connected on the path via a subpath of length at least 2 using only cells in the bands $B_1 - B_i$. In this case we can create a new subpath of the same length that just uses cells in the band B_{i+1} as inner vertices and leading to a new cell c_2'. Because the number of cells doubles in each band this new subpath must cover more cells than the old one and thus we can create a path of length ℓ that covers more cells, a contradiction.

So the optimal path has the desired form, as shown in Figure 3.6. Note that each cell in band i covers 2^{i-1} cells in band B_1. If ℓ is odd, this yields $\frac{\ell-1}{2}$ upward/downward edges each and one edge (2 cells) staying at the same level, yielding $\sum_{i=1}^{(\ell-1)/2} 2^{i-1} + 2 \cdot 2^{(\ell-1)/2} = 3 \cdot 2^{(\ell-1)/2} - 1 < 2^{\ell/2+2}$ covered cells. If ℓ is even, the path uses $\frac{\ell}{2}$ upward/downward edges each and no edge (one cells) staying at the same level, yielding $\sum_{i=1}^{\ell/2} 2^{i-1} + 2^{\ell/2} = 2^{\ell/2+1} - 1 < 2^{\ell/2+2}$ covered cells.

$\square$

Let ℓ_i be the length of the red path connecting c_i' and r_i. We define independent random variables K_i distributed as $\mathsf{Geom}(1 - 8e^{-\nu/5\pi})$. Note that this independence can only make the bound larger as no two of the corresponding paths can meet, so a path cannot use anything that has been exposed already by a different path. Also, the number of available next cells in any step can only go down if we consider the

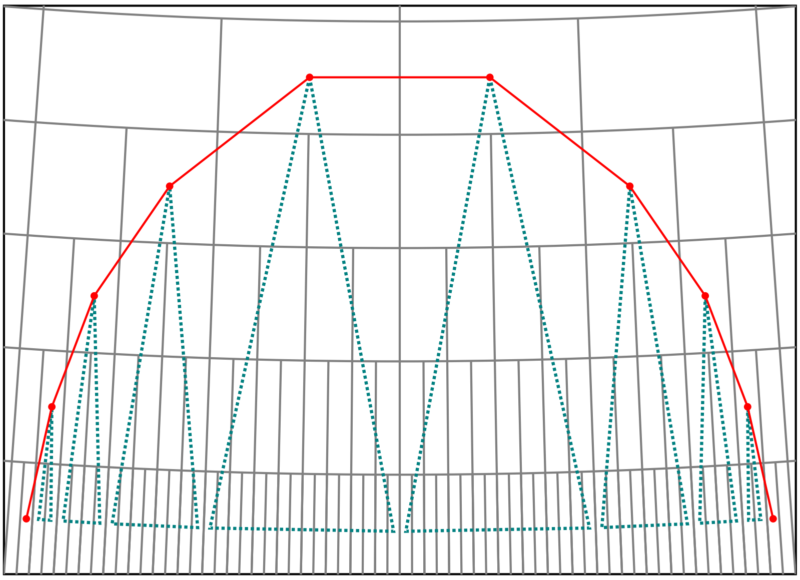

Figure 3.6: Red path of length 9 covering the maximum number of cells.

dependent case. Because every red cell has at most 8 neighbours and every cell is red with probability at most $e^{-\nu/5\pi}$, we have

$$\mathbb{P}(\ell_i \geq K_i) < \left(8e^{-\nu/5\pi}\right)^{K_i}.$$

In other words, ℓ_i is stochastically bounded from above by K_i. Also, by Claim 3.3.8(ii) R_i is stochastically bounded by $2^{\ell_i/2+2}$, which in turn is stochastically bounded from above by $2^{K_i/2+2}$. We denote the latter by Y_i.

We define independent random variables X_i and Y_i, where $X_i = \mathsf{Geom}(e^{-\nu/5\pi})$ and $Y_i = 2^{K_i/2+2}$. With this S_i is stochastically bounded by X_i and R_i is stochastically bounded by Y_i. Define

$$L_t = \sum_{i=1}^{t}(X_i + Y_i)$$

and let $\mathcal{T} := \max\{t : L_t < \frac{2\pi}{\nu}N\}$. As the number of cells in B_1 is at least $\frac{2\pi}{\nu}N$ and with Claim 3.3.8, $\mathcal{T}$ is a stochastic lower bound on the number of steps we need to take to cover B_1 with S_i and R_i, which we denote by K. But the latter is also a bound on the vertices of $\mathcal{P}(N;\alpha,\nu)$ which belong to the component that contains the vertices induced by the lollipop. With $\mu_1 = \mathbb{E}(X_1) = e^{\nu/5\pi}$ and $\mu_2 = \mathbb{E}(Y_1)$ we have $\mathbb{E}(L_t) = t(\mu_1 + \mu_2)$.

Let $\hat{t} = \frac{1.9\pi}{\nu(\mu_1+\mu_2)}N$. Note that $\mu_1 + \mu_2 > 1$ as $\mu_2 \geq 0$ and $\mu_1 = e^{\nu/5\pi} > 1$. Hence, $\mathbb{E}(L_{\hat{t}}) \leq \frac{1.9\pi}{\nu}N$. If $\mathbb{P}(L_{\hat{t}} > \frac{2\pi}{\nu}N) = o(1)$, it follows that a.a.s. $\mathcal{T} > \hat{t}$. Hence, by the stochastic domination we deduce that a.a.s. $K > \hat{t}$.

Claim 3.3.9. *Let $\nu > 20\pi$. We have*

$$\mathbb{P}\left(L_{\hat{t}} > \frac{2\pi}{\nu}N\right) = o(1).$$

Proof. We have

$$\begin{aligned}
\mathrm{Var}(L_t) &= t(\mathrm{Var}(X_1) + \mathrm{Var}(Y_1)),\\
\mathrm{Var}(X_1) &= \mathrm{Var}(\mathsf{Geom}(e^{-\nu/5\pi})) = \frac{1-e^{-\nu/5\pi}}{e^{-2\nu/5\pi}} < e^{\frac{\nu}{2\pi}} \text{ and}\\
\mathrm{Var}(Y_1) &= \mathbb{E}(Y_1^2) - \mathbb{E}^2(Y_1).
\end{aligned}$$

But

$$\mathbb{E}(Y_1^2) = \sum_{k=1}^{\infty} 2^{k+4}(8e^{-\nu/5\pi})^k \leq 16\sum_{k=1}^{\infty}(16e^{-\nu/5\pi})^k \overset{\nu>24\pi}{<} 16\sum_{k=1}^{\infty}(2/e)^k < \frac{32}{e-2} = 64.$$

Thus $\mathrm{Var}(Y_1) < 64$ and $\mathrm{Var}(L_{\hat{t}}) < (64 + e^{\frac{\nu}{2\pi}})\hat{t} = O(N)$. By Chebyshev's inequality,

$$\mathbb{P}\left(L_{\hat{t}} > \frac{2\pi}{\nu}N\right) = O(1/N).$$

$\square$

Note that $\mathbb{E}(X_1) = e^{\frac{\nu}{5\pi}}$ and

$$\mathbb{E}(Y_1) \leq 1 + \sum_{k=1}^{\infty}\left(8e^{-\nu/5\pi}\right)^k 2^{k/2+2} \overset{\nu>20\pi}{<} 1 + 4\sum_{k=1}^{\infty}\left(8e^{-4}\right)^k 2^{k/2} < 1 + 4\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k < 3.$$

So $\hat{t} > \frac{1.9\pi}{\nu(e^{\nu/5\pi}+3)}N$. We can now prove the second part of the main theorem.

Proof of the second part of Theorem 1.6.2. Let $\nu > 20\pi$ and let G_c be defined as above.

By Claim 3.3.7 there is a blue lollipop a.a.s. By Claims 3.3.8.1 and 3.3.9 a.a.s. the blue lollipop extends into a blue component of order at least $\hat{t} > \frac{1.9\pi}{\nu(e^{\nu/5\pi}+3)}N$. Setting $\nu = 20\pi$, this quantity is at least $\frac{1.9\pi}{20\pi(e^4+3)}N > N/610$. Note that increasing ν can only stochastically increase the order of the largest connected component. Thus, for any such $\nu > 20\pi$, the order of the largest component is a.a.s. at least $N/610$.

$\square$

Chapter 4

Connectivity

In the next section, we spell out the short proofs of parts 1 and 2 of Theorem 1.6.3. As mentioned earlier, part 1 follows directly from one of the main results in [GPP12]. It turns out that when $\alpha \leq 1/2$ the probability that the graph is connected can be well approximated by the probability that there exists a set of vertices with small radii such that all of the disk $\mathcal{D}_R$ is covered by the disks of radius R around each of the points. We will call such a set of points a *cover*. In the case when $\alpha < 1/2$ it is relatively easy to show that a cover exists with probability $1 - o(1)$. In the case when $\alpha = 1/2$ determining the probability of the existence of a cover is much more involved. It turns out that this probability can be described in terms of a time-inhomogeneous branching process with infinitely many types.

In the next section we give the quick derivations of parts 1 and 2 of Theorem 1.6.3. In Section 4.2, we review and extend some classical results on multitype branching processes that will be needed in the sequel. In Section 4.3, we describe an auxiliary random process that will help us to derive expressions for the probability of the graph being connected in the case when $\alpha = 1/2$. Finally, in Section 4.4, we derive part 3 of Theorem 1.6.3 from the results in Section 4.3.

4.1 Proof of Theorem 1.6.3: parts 1 and 2

Part 1 is a direct corollary of Theorem 2.2 in [GPP12], since this theorem implies that when $\alpha > \frac{1}{2}$ there are isolated vertices a.a.s.

For Part 2 of Theorem 1.6.3 we argue as follows. Let us fix an arbitrary $\nu > 0$ and $0 < \alpha < \frac{1}{2}$. We partition the disk or radius one around the origin into eight equal slices $S_i := \{(r,\theta) : 0 \leq r \leq 1, (i-1)\pi/4 \leq \theta \leq i\pi/4\}, i = 1, \dots, 8$. Now we let E denote the event that each S_i contains at least one point. See Figure 4.1 for a depiction of the event E. Note that we have

$$\begin{aligned}
\mathbb{P}(E) &\geq 1 - 8 \cdot \left(1 - \tfrac{\cosh(\alpha)-1}{8(\cosh(\alpha R)-1)}\right)^N \\
&\geq 1 - 8 \exp\left[-N \cdot \tfrac{\cosh(\alpha)-1}{8(\cosh(\alpha R)-1)}\right] \\
&\geq 1 - \exp[-\Omega(e^{(1/2-\alpha)R})] \\
&= 1 - o(1),
\end{aligned}$$

where the asymptotics are as $N \to \infty$ (and hence also $R \to \infty$). In the third line we have used that $N = \nu e^{R/2}$ and $\cosh(\alpha R) \sim \frac{1}{2} e^{\alpha R}$ as $N \to \infty$.

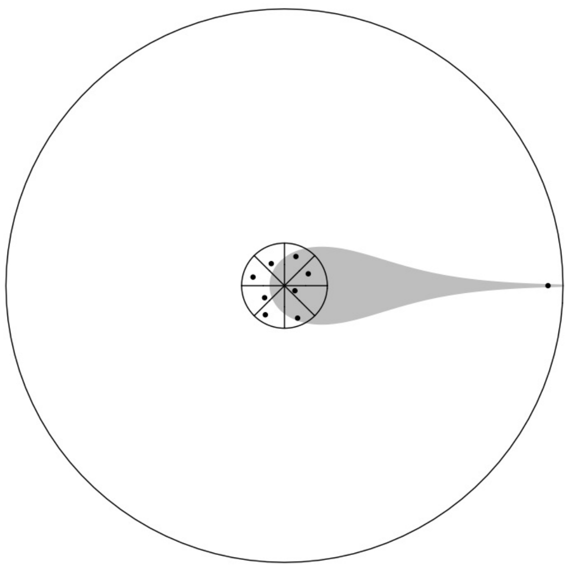

Figure 4.1: The event E, depicted in the native model of the hyperbolic plane. The shaded area shows the disk of radius R around the point on the right.

It remains to show that the event E implies that the graph is connected. To see this, suppose that E holds and let $u = (r, \theta) \in \mathcal{D}_R$ be arbitrary. Then there is a point $v = (r', \theta') \in V_N$ with $|\theta' - \theta| \leq \pi/4$. Let us write $u' := (R, \theta)$ and $v' := (1, \theta')$. Now observe that

$$\begin{aligned}
&\cosh(1)\cosh(R) - \cos(|\theta' - \theta|)\sinh(1)\sinh(R) \\
&\leq \cosh(1)\cosh(R) - \cos(\pi/4)\sinh(1)\sinh(R) \\
&\sim \left((\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2})e + (\tfrac{1}{2} + \tfrac{1}{4}\sqrt{2})e^{-1}\right)\cosh(R),
\end{aligned}$$

where we used that $\cosh R \sim \sinh R$ as $R \to \infty$ in the last line. Since $(\frac{1}{2} - \frac{1}{4}\sqrt{2})e + (\frac{1}{2} + \frac{1}{4}\sqrt{2})e^{-1} < 1$, it follows from the hyperbolic law of cosines that $\mathrm{dist}_{\mathbb{H}}(u', v') \leq R$ provided that N is sufficiently large.

In our case, this last lemma gives that also $\mathrm{dist}_{\mathbb{H}}(u, v) \leq R$, provided N is sufficiently large. As $u \in \mathcal{D}_R$ was arbitrary, it follows that – when N is sufficiently large – the event E implies that every point of $\mathcal{D}_R$ is within distance R of a point of $V \cap B(0, 1)$. Hence, if E holds then the graph is certainly connected – in fact it will have diameter at most three.

This shows that when $0 < \alpha < \frac{1}{2}$ and $\nu > 0$ the graph is a.a.s. connected as claimed in part 2 of Theorem 1.6.3. We now proceed with the proof of part 3.

4.2 Multitype Galton-Watson processes

In preparation for the proof of part 3 of Theorem 1.6.3, we will review and adapt a classical result on Galton-Watson branching processes with finitely many types. If there are $t < \infty$ types, then such a process is described by a sequence $Z_0, Z_1, \ldots$ of random vectors, where $Z_n := (Z_n^1, \ldots, Z_n^t)$ denotes the vector of the number of particles (individuals) of each type in the n-th generation. In each generation, each of the particles replaces itself with a random set of “children", independently of all other particles and the previous history of the process and according to a probability distribution that does not depend on the generation (but it typically does depend on the type of the particle). We denote

$$p(i; z_1, \dots, z_t) := \mathbb{P}(Z_1 = (z_1, \dots, z_t) | Z_0 = e_i).$$

Here and in the rest of the paper e_i denotes the i-th standard basis vector, i.e. the vector with a one in the i-th coordinate and zeroes everywhere else. That is, $p(i; z_1, \dots, z_t)$ is the probability that a particle of type i fathers z_1 children of type 1, z_2 children of type of type 2, and so on until type t. We will say that "extinction" occurs if $Z_n = (0, \dots, 0)$ for some n. Otherwise we say "survival" occurs.

We also set

$$m_{ij} := \mathbb{E}(Z_1^j | Z_0 = e_i).$$

That is, m_{ij} is equal to the expected number of children of type j of a particle of type i; and we write $M := (m_{ij})_{1 \leq i,j \leq t}$ for the "matrix of first moments". Let us also remark that, for every $k \in \mathbb{N}$ and $1 \leq i, j \leq t$ we have $(M^k)_{ij} = \mathbb{E}(Z_k^j | Z_0 = e_i)$ (the expected number of type-j particles in the k-th generation if we start with a single particle of type i). We say that the process is *positive regular* if there exists a $k \in \mathbb{N}$ such that every entry of M^k is positive. By the Perron-Frobenius theorem a positive regular matrix has a real, positive eigenvalue ρ that is larger in absolute value than all other eigenvalues (see for instance [Har63], Chapter II, Section 5, page 37). A multitype Galton-Watson process is called *singular* if each particle has exactly one child (with probability one). Otherwise it is *non-singular*.

A proof of the following standard result can for instance be found in the book by Harris [Har63] (Theorem 7.1, Chapter II, page 41), who attributes it to Sevast'yanov [Sev48] and independently Everett and Ulam [EU].

Theorem 4.2.1. *Consider a positive regular, non-singular multitype Galton-Watson* *process with finitely many types, and let ρ denote the largest eigenvalue of its first moment matrix M. Then the following hold:*

1. *If $\rho \leq 1$ then $\mathbb{P}(extinction|Z_0 = e_i) = 1$ for all types $1 \leq i \leq t$;*

2. *If $\rho > 1$ then $\mathbb{P}(extinction|Z_0 = e_i) < 1$ for all types $1 \leq i \leq t$.*

If $Z_0, Z_1, \dots$ is as in Theorem 4.2.1 and ρ is the largest eigenvalue of M then we say the process is *subcritical* if $\rho < 1$, we say it is *critical* if $\rho = 1$ and we say it is *supercritical* if $\rho > 1$.

The following straightforward observation will be used in the sequel. For completeness we spell out a short proof.

Lemma 4.2.2. *Suppose that $Z_0, Z_1, \dots$ is a positive regular, non-singular, supercritical Galton Watson process with $t < \infty$ types. Then there exists another t-type Galton-Watson process $Y_0, Y_1, \dots$ such that*

1. *$p_Y(i; z_1, \dots, z_t) = 0$ if $p_Z(i; z_1, \dots, z_t) = 0$;*

2. *$p_Y(i; z_1, \dots, z_t) < p_Z(i; z_1, \dots, z_t)$ if $p_Z(i; z_1, \dots, z_t) > 0$ and $(z_1, \dots, z_t) \neq (0, \dots, 0)$;*

and Y is positive regular, non-singular and supercritical.

Proof. Let us fix a $0 < \delta < 1$, to be made specific later, and let us define the offspring distributions of Y by:

$$p_Y(i; z_1, \dots, z_t) = \begin{cases} (1-\delta) \cdot p_Z(i; z_1, \dots, z_t) & \text{if } (z_1, \dots, z_t) \neq (0, \dots, 0), \\ p_Z(i; 0, \dots, 0) + \delta \cdot (1 - p_Z(i; 0, \dots, 0)) & \text{if } (z_1, \dots, z_t) = (0, \dots, 0). \end{cases}.$$

It is easy to see that this way Y is non-singular and that $m_{ij}^Y = (1-\delta) m_{ij}^Z$. So in particular Y is also positive regular, and the largest eigenvalue of its first moment matrix satisfies $\rho_Y = (1-\delta)\rho_Z$. Hence we can choose δ so that $\rho_Y > 1$, in which case Y is as required. $\square$

Let us say that a Galton-Watson process $Z_0, Z_1, \ldots$ *stochastically dominates* a process $Y_0, Y_1, \ldots$ if there is a coupling such that $Z_n^i \geq Y_n^i$ for all $n \in \mathbb{N}$ and all types i. (Note that if the two processes do not have the same number of types then we can formally add types to the one with fewer types and redefine the offspring distributions in such a way that no particle ever gives birth to a child of the new types.) It is for instance easily seen that the process Y from the previous lemma is stochastically dominated by the original process Z.

We say that *explosion* occurs, if the total number of particles grows without bounds. In other words,

$$\{\text{explosion}\} = \left\{ \lim_{n\to\infty} \left(Z_n^1 + \cdots + Z_n^t\right) = \infty \right\}.$$

If $Z_0, Z_1, \ldots$ is as in Theorem 4.2.1 above, then Theorem 6.1 on page 39 of [Har63] states that for every vector $z = (z_1, \ldots, z_t)$ other than the all-zero vector there are only finitely many generations n for which $Z_n = z$ (with probability one). This has the following immediate corollary.

Theorem 4.2.3. *If $Z_0, Z_1, \ldots$ is a positve regular, non-singular multitype Galton-Watson process with finitely many types, then*

$$\mathbb{P}(\textit{extinction}|Z_0 = z) + \mathbb{P}(\textit{explosion}|Z_0 = z) = 1,$$

for every initial state z.

It is natural to also consider multitype Galton-Watson processes with countably many types. In this case the state of the i-th generation is of course a random vector $Z_i = (Z_i^1, Z_i^2, \ldots)$ of countably many nonnegative numbers. We define $p(i; z_1, z_2, \ldots)$ and m_{ij} analogously to the case of finitely many types. For $t \in \mathbb{N}$, the t-restriction of a Galton-Watson process $Z_0, Z_1, \ldots$ with countably many types is the t-type Galton-Watson process $Y_0, Y_1, \ldots$ with offspring distributions given by:

$$p_Y(i; z_1, \dots, z_t) := \begin{cases} p_Z(i; z_1, \dots, z_t, 0, 0, \dots) & \text{if } (z_1, \dots, z_t) \neq (0, \dots, 0), \\ 1 - \sum_{(z_1,\dots,z_t)\neq(0,\dots,0)} p_Y(i; z_1, \dots, z_t) & \text{if } (z_1, \dots, z_t) = (0, \dots, 0). \end{cases}$$

That is, the probability that a particle of type i in the Y process has z_1 children of type 1, z_2 children of type 2 and so on up to type t, is the probability the a particle of type i under the Z process has exactly these children and none of type bigger than t. We can think of the t-restricted process as a version of the old process where a particle and its potential children die during labour if at least one of the potential children has a type $> t$.

Observe that the original process stochastically dominates the t-restricted process.

Lemma 4.2.4. *Suppose $Z_0, Z_1, \dots$ is a multitype Galton-Watson process with countably many types, that satisfies the following conditions:*

1. *There exists a $c > 1$ such that, for every $i \in \mathbb{N}$, we have $\sum_{j=1}^{\infty} j \cdot m_{ij} \geq c \cdot i$;*
2. *For every $i \in \mathbb{N}$ and $j \leq 2i$ we have $m_{ij} > 0$;*
3. *Whenever $p(i; z_1, z_2, \dots) > 0$ we have $\sum_{j=1}^{\infty} j \cdot z_j \leq 2i$. (for every $i \in \mathbb{N}, z_1, z_2, \dots \geq 0$);*
4. *We have*

$$\lim_{i\to\infty} \sum_{\substack{z_1,z_2,\dots\geq 0,\\ z_{i+1}+z_{i+2}+\dots>0}} p(i; z_1, z_2, \dots) = 0.$$

(That is, the probability that a particle of type i has at least one child of a strictly larger type is small for large i.)

Then there exists a $t \in \mathbb{N}$ such that the t-restricted process is positive regular, non-singular and supercritical.

Proof. Observe that, by part 2, the t-restricted process is positive regular and non-singular for every $t \geq 1$. Let $\varepsilon > 0$ be arbitrary, to be determined later. By part 4, there exists a t_0 such that the probability that a particle of type $i \geq t_0$ has a child of type greater than i amongst its children is at most ε. That is:

$$\sum_{\substack{z_1,z_2,\dots\geq 0,\\ z_{i+1}+z_{i+2}+\dots>0}} p(i;z_1,z_2,\dots) < \varepsilon \quad \text{(for all } i \geq t_0).$$

We now set $t := 2t_0$. Then we have that

$$\sum_{\substack{z_1,z_2,\dots\geq 0,\\ z_{t+1}+z_{t+2}+\dots>0}} p(i;z_1,z_2,\dots) = 0 \quad \text{if } i < t_0,$$

by condition 3 of the lemma. And, if $t_0 \leq i \leq t$ then we have:

$$\sum_{\substack{z_1,z_2,\dots\geq 0,\\ z_{t+1}+z_{t+2}+\dots>0}} p(i;z_1,z_2,\dots) \leq \sum_{\substack{z_1,z_2,\dots\geq 0,\\ z_{i+1}+z_{i+2}+\dots>0}} p(i;z_1,z_2,\dots) < \varepsilon. \tag{4.1}$$

Let $M = (m_{ij})_{i,j\geq 1}$ denote the matrix of first moments of the original process, and let $M' = (m'_{ij})_{1\leq i,j\leq t}$ denote that of the t-restricted process. We have that, for every $1 \leq i \leq t$:

$$\sum_{j=1}^{t} j \cdot m'_{ij} \geq \sum_{j=1}^{\infty} j \cdot m_{ij} - \varepsilon \cdot 2i \geq (c - 2\varepsilon)i,$$

using conditions 1, 3 of the lemma and (4.1). Thus, if we chose ε small enough so that $c' := c - 2\varepsilon > 1$, then we see that if $v := (1, 2, \dots, t)$ then $(M')^k v \geq (c')^k v$ coordinatewise. Since $(c')^k$ grows without bounds, it follows that M' must have an eigenvalue that is strictly larger than one in absolute value. So in particular (invoking Perron-Frobenius) the eigenvalue of largest absolute value is a real number strictly larger than one. This concludes the proof of the lemma. $\square$

In a time-inhomogeneous multitype Galton-Watson process, the offspring distibutions depend on n, the generation. We now denote by $p_n(i;z_1,z_2,\dots) := \mathbb{P}(Z_{n+1} =$

$(z_1, z_2, \dots)|Z_n = e_i)$ the probability that a particle of type i, in generation n, fathers exactly z_j children of type j (for $j = 1, 2, \dots$).

Lemma 4.2.5. *Suppose that $Z_0, Z_1, \dots$ is a time-inhomogeneous multi-type Galton-Watson process with countably many types such that the limits*

$$\lim_{n\to\infty} p_n(i; z_1, z_2, \dots) =: p(i; z_1, z_2, \dots),$$

exist for all $i \in \mathbb{N}$ and $z_1, z_2, \dots \geq 0$. Suppose further that the limits p belong to a (time-homogeneous) multitype Galton-Watson process satisfying the conditions of Lemma 4.2.4. Then

$$\liminf_{n\to\infty} \mathbb{P}(\textit{explosion}|Z_n = e_1) > 0.$$

Proof. Let $Z_0', Z_1', \dots$ denote the Galton-Watson process belonging to the limiting probabilities $p(i; z_1, z_2, \dots)$ and let us pick t according to Lemma 4.2.4 with respect to Z'. Let $Y_0, Y_1, \dots$ denote the t-restricted process.

Let $X_0, X_1, \dots$ denote a process that Lemma 4.2.2 provides if we apply it to $Y_0, Y_1, \dots$. Let $\mathcal{I} := \{(z_1, \dots, z_t) \neq (0, \dots, 0) : p_X(i; z_1, \dots, z_t) > 0\}$. Observe that $\mathcal{I}$ is finite, so that there is an n such that $p_{n+m}(i; z_1, \dots, z_t, 0, 0, \dots) \geq p_X(i; z_1, \dots, z_t)$, for all $m \geq 0$ and all $(z_1, \dots, z_t) \in \mathcal{I}$. This means that $Z_n, Z_{n+1}, \dots$ stochastically dominates $X_0, X_1, \dots$, if we condition on $Z_n = X_0 = e_1$. So in particular:

$$\liminf_{n\to\infty} \mathbb{P}(Z \text{ explodes}|Z_n = e_1) \geq \mathbb{P}(X \text{ explodes}|X_0 = e_1) > 0.$$

This concludes the proof of the lemma. □

4.3 An auxiliary coverage process

In this section, we consider an auxiliary random process that is closely related to the KPKVB random graph with $\alpha = 1/2$. In the rest of the paper, $\mathcal{P} = \mathcal{P}_\nu$ will be a Poisson process on the *entire* hyperbolic plane with intensity function:

$$g\big(r,\theta\big) = g_\nu\big(r,\theta\big) := (\nu/4\pi)\cdot\sinh(r/2), \tag{4.2}$$

where (r,θ) represents a point of $\mathbb{H}$ in polar coordinate notation. We let $\mathbb{P}_\nu(.)$ denote the associated probability measure. $\mathbb{E}_\nu(.)$ denotes the expected values of random variables over the probability space. We say that an event $E(N)$ is realized with high probability (w.h.p.), if $\mathbb{P}_\nu(E(N)) \to 1$ as $n \to \infty$.

We set

$$\gamma(r) = \gamma_\lambda(r) := \lambda\cdot\arccos\Big(\frac{\cosh(r)-1}{\sinh(r)}\Big), \tag{4.3}$$

where $\lambda > 0$ is a parameter. We will see in the proof of Lemma 4.4.2 that if two points $x_1 = (r_1,\theta_1), x_2 = (r_2,\theta_2) \in \mathcal{D}_R$ have $|\theta_1-\theta_2|_{2\pi} \le \gamma(r_1)$ with $\lambda < 1$, then x_1 and x_2 are within distance R (provided N is large). Here and in the rest of the paper we use the notation $x_r := \min(x, r-x)$ for $r > 0$ and $x \in [0,r]$. Let us remark that $\gamma(r)$ is strictly decreasing in r. (This can be easily seen from the facts that arccos(.) is strictly decreasing and that $(\cosh(r)-1)/\sinh(r) = 1-\frac{2}{e^r+1}$ is strictly increasing.) Let us say that an angle $\vartheta \in [0,2\pi)$ is *covered* by a point $(r,\theta)\in\mathbb{H}$ if

$$|\vartheta-\theta|_{2\pi} \le \gamma_\lambda(r).$$

We say that a set $A \subseteq \mathbb{H}$ is a *cover* if every angle is covered by some point of A. For $s > 0$, we denote by $\mathcal{C}_s(\lambda)$ the event that $\mathcal{P}\cap B_{\mathbb{H}}(O,s)$ is a cover. The event $\mathcal{C}(\lambda)$ will denote that $\mathcal{C}_s(\lambda)$ is realized for some (finite) $s < \infty$. Note that $\mathcal{C}(\lambda) = \bigcup_{s>0}\mathcal{C}_s(\lambda)$. We now define:

$$\Psi(\nu,\lambda) := \mathbb{P}_\nu(\mathcal{C}(\lambda)). \tag{4.4}$$

As we will see, $f(\nu) := \Psi(\nu,1)$ has the properties claimed in Theorem 1.6.3 and the probability that $G(N;1/2,\nu)$ is connected tends to $f(\nu)$ as $N\to\infty$. The following theorem is crucial for the proof of part 3 of Theorem 1.6.3.

Theorem 4.3.1. *The function Ψ defined in* (4.4) *has the following properties:*

1. $\Psi(\nu, \lambda)$ *is continuous in both parameters;*
2. $\Psi(\nu, \lambda) = 1$ *if* $\nu \cdot \lambda \geq \pi$*;*
3. $\Psi(\nu, \lambda)$ *is strictly increasing in* ν *for* $0 < \nu < \pi/\lambda$*;*
4. *For every fixed* $\lambda > 0$ *we have* $\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) = 0$.

The remainder of this section is devoted to the rather involved proof of this theorem. We will split the proof up into a sequence of lemmas.

Lemma 4.3.2. $\Psi(\nu, \lambda) > 0$ *for all* $\nu, \lambda > 0$.

Proof. Let us set $m := \min\{4, \lceil 4/\lambda \rceil\}$, and let E be the event that each of the $2m$ sets $[0, \pi/m) \times [0, 1], \ldots, [(2m-1)\pi/m, 2\pi) \times [0, 1]$ contains at least one point of $\mathcal{P}$. The expected number of points of $\mathcal{P}$ in each of these sets is $\frac{1}{2m} \cdot \int_0^{2\pi} \int_0^1 g(r, \theta) \mathrm{d}r \mathrm{d}\theta = (\nu/2m) \cdot (\cosh(1/2) - 1)$.

It is easily checked that $\arccos((\cosh(1) - 1)/\sinh(1)) > \pi/4$, so that $\gamma(r) > \lambda\pi/4$ for all $r \leq 1$. We claim the event E implies $\mathcal{C}(\lambda)$. To see this, suppose E is realized and pick an arbitrary angle $\theta \in [0, 2\pi)$. By symmetry, we can assume without loss of generality $\theta \in [0, \pi/m)$. Since E holds, there is a point $(r, \vartheta) \in \mathcal{P} \cap [0, \pi/m) \times [0, 1]$. We find that $|\theta - \vartheta|_{2\pi} < \pi/m \leq \lambda\pi/4 < \gamma(r)$. Thus, the event E indeed inplies $\mathcal{C}(\lambda)$.

We therefore have

$$\Psi(\nu, \lambda) \geq \mathbb{P}_\nu(E) = \left(1 - e^{-(\nu/m) \cdot (\cosh(1/2) - 1)}\right)^m > 0,$$

as required. □

Lemma 4.3.3. *For all* $a, b, \lambda > 0$ *we have* $\Psi(a+b, \lambda) \geq \Psi(a, \lambda) + (1 - \Psi(a, \lambda)) \cdot \Psi(b, \lambda)$.

Proof. Since $\mathcal{P}_{a+b}$ can be seen as a *superposition* of $\mathcal{P}_a$ and $\mathcal{P}_b$ for every $a, b > 0$ (see for instance [Kin93]), the probability that $\mathcal{C}(\lambda)$ occurs in $\mathcal{P}_{a+b}$ is at least the probability that it occurs in $\mathcal{P}_a$ plus the probability that is does not occur in $\mathcal{P}_a$ and it occurs in $\mathcal{P}_b$. $\square$

Note that the previous two lemmas show that $\Psi(\nu, \lambda)$ is strictly increasing in ν whenever $\Psi(\nu, \lambda) < 1$.

Corollary 4.3.4. *If $\nu, \lambda > 0$ are such that $\Psi(\nu, \lambda) < 1$ then Ψ is strictly increasing in ν at (ν, λ).*

It will be helpful to consider a process where we reveal $\mathcal{P}$ in "discrete steps". For $n \in \mathbb{N}$ let us denote

$$r_n := n \cdot 2 \ln 2. \tag{4.5}$$

Let us denote $\mathcal{B}_n := \mathcal{P} \cap B_{\mathbb{H}}(0, r_n)$ and $\mathcal{A}_n := \mathcal{B}_n \setminus \mathcal{B}_{n-1}$. ($\mathcal{B}_n$ is the set of points of $\mathcal{P}$ with radii at most r_n and $\mathcal{A}_n$ is the set of points with radii between r_{n-1} and r_n.)

Before we continue, it will be helpful to derive some asymptotics. Observe that

$$\frac{\cosh(r) - 1}{\sinh(r)} = 1 - 2e^{-r} \left(\frac{1 - e^{-r}}{1 - e^{-2r}} \right). \tag{4.6}$$

Recall that $\cos(y) = 1 - y^2/2 + O(y^4)$. This implies that if $y = \arccos(1 - x)$ then $y = \sqrt{2x} \cdot (1 + O(x^2))$. Combining this with (4.6) gives:

$$\gamma(r) = \lambda \cdot \arccos\left(\frac{\cosh(r) - 1}{\sinh(r)} \right) = 2\lambda e^{-r/2}(1 + O(e^{-r})) \quad \text{as } r \to \infty. \tag{4.7}$$

Let us also recall that $\gamma(r)$ is strictly decreasing in r. (As $(\cosh(r) - 1)/\sinh(r) = 1 - 2/(e^r + 1)$ is strictly increasing and $\arccos(.)$ is strictly decreasing.) Using equation (4.7) we can now derive the following.

Lemma 4.3.5. *For every fixed $\nu, \lambda > 0$ we have that*

$$\mathbb{E}_\nu|\{p \in \mathcal{A}_n : p \text{ covers the angle } 0\}| = (1 + O((1/4)^n)) \cdot (\nu\lambda/\pi) \cdot \ln 2,$$

and

$$\mathbb{P}_\nu(\mathcal{A}_n \text{ does not cover } 0) = (1 + O((1/4)^n)) \cdot (1/2)^{\nu\lambda/\pi},$$

where the $O(.)$-notation refers to $n \to \infty$.

Proof. If μ_n denotes the expected number of points in $\mathcal{A}_n$ that cover the angle 0, then

$$\begin{aligned} \mu_n &= \int_0^{2\pi} \int_{r_{n-1}}^{r_n} 1_{\{|\theta|_{2\pi} < \gamma(r)\}} \cdot g(r,\theta) \mathrm{d}r \mathrm{d}\theta \\ &= \int_{r_{n-1}}^{r_n} 2\gamma(r) \cdot g(r,\theta) \mathrm{d}r \\ &= \int_{r_{n-1}}^{r_n} 4\lambda(1 + O(e^{-r}))e^{-r/2} \cdot (\nu/4\pi) \cdot \sinh(r/2) \mathrm{d}r \\ &= \int_{r_{n-1}}^{r_n} 4\lambda(1 + O(e^{-r}))e^{-r/2} \cdot (\nu/4\pi) \cdot (1 + O(e^{-r}))\frac{1}{2}e^{r/2} \mathrm{d}r \\ &= (1 + O(e^{-r_n})) \cdot (\nu\lambda/2\pi) \int_{r_{n-1}}^{r_n} 1 \, \mathrm{d}r \\ &= (1 + O(4^{-n})) \cdot (\nu\lambda/\pi) \cdot \ln 2. \end{aligned} \tag{4.8}$$

Here we used that $\sinh(x) = (1 + O(e^{-x})) \cdot \frac{1}{2}e^x$ for large x. This proves the first statement of the lemma. The second statement follows immediately from the fact that $\mathbb{P}_\nu(\mathcal{A}_n \text{ covers } 0) = e^{-\mu_n}$ (Fact 2.2.3). □

Lemma 4.3.6. *We have $\gamma(r_n) > \lambda \cdot 2^{-n}$, for all $n \in \mathbb{N}$.*

Proof. It suffices to prove that

$$\varphi(r) := e^{r/2} \cdot \gamma(r)/\lambda = e^{r/2} \cdot \arccos\left(\frac{\cosh(r) - 1}{\sinh(r)}\right),$$

is strictly larger than one for all $r \geq r_1 = 2\ln 2$. Observe that $\cos(y) \geq 1 - y^2/2$ for all $y \in \mathbb{R}$. This implies that if $y = \arccos(1-x)$ then $y \geq \sqrt{2x}$. Combining this with (4.6) shows that

$$\varphi(r) \geq e^{r/2} \cdot 2e^{-r/2} \left(\frac{1-e^{-r}}{1-e^{-2r}}\right)^{1/2} = 2\left(\frac{1-e^{-r}}{1-e^{-2r}}\right)^{1/2} \geq 2\sqrt{1-e^{-r}} \geq \sqrt{3} > 1,$$

using that $r \geq 2\ln 2$ for the penultimate inequality. □

Lemma 4.3.7. *For every $\nu, \lambda > 0$ there exists a $c = c(\nu, \lambda) > 0$ such that*

$$\mathbb{P}_\nu\big[\mathcal{A}_n \text{ covers } [0, \lambda 2^{-n})\big] \geq c,$$

(i.e., the probability that $[0, \lambda 2^{-n})$ is covered in its entirety by the points of $\mathcal{P}$ with radii between r_{n-1} and r_n is at least c) for all $n \in \mathbb{N}$.

Proof. It follows from Lemma 4.3.6 and the monotonicity of $\gamma(r)$ that if (r, θ) covers 0 and furthermore $\theta \in [0, \pi)$ and $r \leq r_n$ then (r, θ) in fact covers all of $[0, \lambda 2^{-n})$. It follows that

$$\begin{aligned}\mathbb{P}_\nu\big[\mathcal{A}_n \text{ covers } [0, \lambda 2^{-n})\big] &\geq \frac{1}{2} \cdot \mathbb{P}_\nu\big[\mathcal{A}_n \text{ covers } 0\big] \\ &= \frac{1}{2} \cdot \big(1 - (1 + O((1/4)^n)) \cdot (1/2)^{\nu\lambda/\pi}\big) = \Omega(1),\end{aligned}$$

using Lemma 4.3.5. □

Let us write $\mathcal{U}_n \subseteq [0, 2\pi)$ for the union of intervals of angles *not* covered by the points of $\mathcal{B}_n$. Then $\mathcal{U}_n$ clearly consists of a finite number of intervals. Let $\mathcal{U}_n^{\text{long}} \subseteq \mathcal{U}_n$ denote the union of all intervals of length at least $\lambda 2^{-n}$, and let $\mathcal{U}_n^{\text{short}} := \mathcal{U}_n \setminus \mathcal{U}_n^{\text{long}}$ denote the union of all intervals strictly shorter than $\lambda 2^{-n}$.

We now also define

$$L_n = L_n(\lambda) := \text{length}(\mathcal{U}_n) \cdot \lambda^{-1} \cdot 2^n, \quad L_n^{\text{long}} = L_n^{\text{long}}(\lambda) := \text{length}(\mathcal{U}_n^{\text{long}}) \cdot \lambda^{-1} \cdot 2^n,$$
$$L_n^{\text{short}} = L_n^{\text{short}}(\lambda) := \text{length}(\mathcal{U}_n^{\text{short}}) \cdot \lambda^{-1} \cdot 2^n. \quad (4.9)$$

The λ is omitted when it is clear from the context.

That is, L_n denotes total length of $\mathcal{U}_n$, multiplied by $\lambda^{-1}2^n$ and $L^{\text{long}}, L^{\text{short}}$ are defined analogously. We let $\mathcal{N}_n^{\text{short}}$ denote the number of components of $\mathcal{U}_n^{\text{short}}$ (i.e. the number of intervals of length strictly less than $\lambda 2^{-n}$), and we set

$$Y_n := \mathcal{N}_n^{\text{short}} + L_n^{\text{long}}. \tag{4.10}$$

Recall that if $(E_n)_n$ is a sequence of events then we say the event "E_n *almost always*" holds if E_n holds for all but finitely many n. In other words $\{E_n \text{ almost always}\} = \liminf E_m = \bigcup_n \bigcap_{m>n} E_m$. We can for instance write

$$\{\mathcal{C}(\lambda)\} = \{L_n = 0 \text{ almost always}\} = \{Y_n = 0 \text{ almost always}\}.$$

Also recall that we say that the event "E_n *infinitely often*" holds if E_n holds for infinitely many n. In other words $\{E_n \text{ infinitely often}\} = \bigcap_n \bigcup_{m>n} E_m$.

Lemma 4.3.8. *For every $\nu, \lambda, K > 0$ we have $\mathbb{P}_\nu(Y_n > K \text{ almost always}) = 1 - \Psi(\nu, \lambda)$.*

Proof. Observe that $\mathbb{P}_\nu(Y_n = 0 \text{ almost always}) = \Psi(\nu, \lambda)$. Let us also observe that, for every $K > 0$:

$$\begin{aligned}\mathbb{P}_\nu(Y_n = 0 \text{ almost always}) + \mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often})& \\ + \mathbb{P}_\nu(Y_n > K \text{ almost always}) = 1.&\end{aligned}$$

Hence, it suffices to show that $\mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often }) = 0$ for every $K > 0$. Observe that if $Y_n = y$, then $\mathcal{U}_n$ can be covered by at most $2\lceil y/\lambda \rceil$ intervals of length $\lambda 2^{-n}$. By Lemma 4.3.7, and positive correlation, there exists a $c > 0$ such that for all $y > 0$:

$$\mathbb{P}_\nu(Y_{n+1} = 0 | Y_n = y, Y_{n-1} = y_{n-1}, \dots, Y_1 = y_1) \geq c^{2\lceil y/\lambda \rceil}, \tag{4.11}$$

for all $n \in \mathbb{N}$ and all $y, y_1, \dots, y_{n-1} > 0$. Now let N_1 be the (random) $n \in \mathbb{N}$ for which

$Y_n \in (0, K]$ for the first time. Similarly, let N_i be the i-th index n for which $Y_n \in (0, K]$. (Here we set $N_i = \infty$ if $Y_n \in (0, K]$ for less than i indices n.) It follows from (4.11) that $\mathbb{P}_\nu(N_{i+1} < \infty | N_i < \infty) \le 1 - c^{2\lceil K/\lambda \rceil} =: x$. But then we also have that, for every $M \in \mathbb{N}$:

$$\begin{aligned} \mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often }) &\le \mathbb{P}_\nu(N_i < \infty \text{ for all } 1 \le i \le M) \\ &= \mathbb{P}_\nu(N_1 < \infty) \cdot \textstyle\prod_{i=1}^{M-1} \mathbb{P}_\nu(N_{i+1} < \infty | N_i < \infty) \\ &\le 1 \cdot x^{M-1}. \end{aligned}$$

Sending $M \to \infty$ shows that $\mathbb{P}_\nu(Y_n \in (0, K]$ infinitely often$) = 0$, as required. □

Lemma 4.3.9. *If $I \subseteq \mathcal{U}_n$ is an interval then $I \cap \mathcal{U}_{n+1}$ consists of at most $\left\lfloor \frac{\text{length}(I)}{\lambda 2^{-n}} \right\rfloor + 1$ intervals.*

Proof. Notice that, if the interval I is cut into $k+1$ disjoint, non-empty intervals by $\mathcal{A}_{n+1}$ then there must be k points $(\rho_1, \theta_1), \dots, (\rho_k, \theta_k) \in A_{n+1}$ such that the intervals $(\theta_i - \gamma(\rho_i), \theta_i + \gamma(\rho_i))$ are disjoint and completely contained in I. Hence we must have that

$$\text{length}(I) > \sum_{i=1}^{k} 2\gamma(r_i) \ge 2k\gamma(r_n) > k\lambda 2^{-n},$$

using Lemma 4.3.6. The lemma follows. □

Corollary 4.3.10. *If $I \subseteq \mathcal{U}_n$ is an interval of length at most $\lambda 2^{-n}$ then $I \cap \mathcal{U}_{n+1}$ is either empty or a single interval.*

Another relatively obvious, but key, observation is the following.

Lemma 4.3.11. *If $I, J \subseteq [0, 2\pi)$ are two sets such that $|x - y|_{2\pi} \ge 2\gamma(r_n)$ for all $x \in I, y \in J$, then $I \cap \mathcal{A}_m$ and $J \cap \mathcal{A}_m$ are independent for all $m > n$.*

Proof. This follows immediately from the fact that a point of radius bigger than r_n cannot simultaneously cover two angles that are more than $2\gamma(r_n)$ apart, and the fact that $\mathcal{P}_\nu \cap A$ and $\mathcal{P}_\nu \cap B$ are independent if $A, B \subseteq \mathbb{H}$ are disjoint. □

Lemma 4.3.12. *For every $\nu, \lambda, K > 0$ we have $\mathbb{P}_\nu(L_n^{long} > K$ infinitely often$) = 1 - \Psi(\nu, \lambda)$.*

Proof. Recall that $\Psi(\nu, \lambda) = \mathbb{P}_\nu(L_n = 0$ almost always$)$. This immediately gives us that $\mathbb{P}_\nu(L_n > 0$ almost always$) = 1 - \Psi(\nu, \lambda)$. It thus suffices to show that, for every $K > 0$, $\mathbb{P}_\nu(L_n > 0$ and $L_n^{\text{long}} < K$ almost always$) = 0$. Suppose that, on the contrary, for some $K > 0$ it holds that

$$\mathbb{P}_\nu(L_n > 0 \text{ and } L_n^{\text{long}} < K \text{ almost always}) > 0.$$

It must then also be the case that $\mathbb{P}_\nu(Y_n > K'$ and $L_n^{\text{long}} < K$ almost always$) > 0$, for every constant K' by Lemma 4.3.8. And, since $Y_n = \mathcal{N}_n^{\text{short}} + L_n^{\text{long}}$, we must then also have that
$\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K'$ and $L_n^{\text{long}} < K$ almost always$) > 0$, for every constant K'. Let us remark that, if E_n almost always holds, then there is a (random) N such that E_n holds for all $n \geq N$. Hence, to prove the lemma it suffices to show that for every $K > 0$ there exists a $K' = K'(K) > 0$ such that $\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K'$ and $L_n^{\text{long}} < K$ for all $n \geq n_0) = 0$, for all $n_0 \in \mathbb{N}$.

Let $K > 0$ thus be arbitrary. Let $c = c(\nu, \lambda)$ be as provided by Lemma 4.3.7, and let us choose K' such that $K' > 8K/c$ and

$$\mathbb{P}(\text{Bi}(a, c) > ac/2) \geq 2/3,$$

for all $a \geq K'$. (The existence of such a K' follows for instance from the Chebyschev bound.)

Observe that, by Lemma 4.3.9, if $L_n^{\text{long}} \leq K$ then the long components (intervals) of generation n will split into no more than $2K$ components in generation $n + 1$. On the other hand, the short intervals of generation n each disappear with probability $\geq c$ and if they don't disappear then they cannot split into two or more intervals by Lemma 4.3.9.

This shows that for all $a \geq K', b \leq K$ we have

$$\mathbb{P}_\nu(\mathcal{N}_{n+1}^{\text{short}} < (1-c/4)\mathcal{N}_n^{\text{short}} | \mathcal{N}_n^{\text{short}} = a, L_n^{\text{long}} = b) \geq 2/3.$$

(To see this note that, with probability 2/3, no more than $(1-c/2)a$ short intervals survive to the next generation, while the long intervals generate at most $2b \leq 2K < K' \cdot c/4 \leq ac/4$ short ones.)

On the other hand, if $\mathcal{N}_n^{\text{short}} = a$ and $L_n^{\text{long}} \leq K$ then a (deteministic) upper bound is $\mathcal{N}_{n+1}^{\text{short}} \leq a + 2cK/8 \leq (1+c/4)\mathcal{N}_n^{\text{short}}$.

Let us now fix arbitrary $n_0 \in \mathbb{N}, a_0 > K', b_0 \leq K$. If $\mathcal{N}_{n_0}^{\text{short}} = a_0, L_{n_0}^{\text{long}} = b_0$ and $\mathcal{N}_n^{\text{short}} > K', L_n^{\text{long}} \leq K$ for all $n \geq n_0$ then, for every $m \geq 2\log(K'/a_0)/\log(1-c^2/16)$, there are more than $m/2$ indices $n \leq i \leq n+m-1$ such that $\mathcal{N}_{i+1}^{\text{short}} > (1-c/4)\mathcal{N}_i^{\text{short}}$. (Otherwise we would have that $\mathcal{N}_m^{\text{short}} < ((1-c/4)(1+c/4))^{m/2} \cdot a_0 = (1-c^2/16)^{m/2} \cdot a_0 < K'$.) Thus, we have

$$\begin{aligned}
&\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K', L_n^{\text{long}} \leq K \text{ for all } n \geq n_0 | \mathcal{N}_{n_0}^{\text{short}} = a_0, L_{n_0}^{\text{long}} = b_0) \\
&\leq \lim_{m\to\infty} \mathbb{P}(\text{Bi}(m, 1/3) \geq m/2) = 0.
\end{aligned}$$

(The last inequality follows for instance from the weak law of large numbers.) Since n_0, a_0, b_0 were arbitrary, it follows that

$$\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K' \text{ and } L_n^{\text{long}} < K \text{ for all } n \geq n_0) = 0 \quad \text{for all } n_0 \in \mathbb{N},$$

as required. □

Lemma 4.3.13. *If $\nu \cdot \lambda = \pi$ then there exists a constant $C = C(\nu, \lambda)$ such that $\mathbb{E}_\nu L_n \leq C$ for all n.*

Proof. For every $\nu, \lambda > 0$, we have that

$$\begin{aligned}\mathbb{E}_\nu L_n &= 2^n \cdot \int_0^{2\pi} \mathbb{P}_\nu(\text{the angle } \theta \text{ is covered by } \mathcal{B}_n)\mathrm{d}\theta \\ &= 2^n \cdot 2\pi \cdot \mathbb{P}_\nu(\text{ the angle } 0 \text{ is covered by } \mathcal{B}_n).\end{aligned}$$

Hence, when $\nu \cdot \lambda = \pi$, we have

$$\mathbb{E}_\nu L_n = 2^n \cdot 2\pi \cdot \exp\left[-\sum_{i=1}^{n}(1+O((1/4)^i)) \cdot \ln 2\right] = 2^n \cdot 2\pi \cdot \exp[-n \ln 2 + O(1)] = O(1),$$

using Lemma 4.3.5. □

Lemma 4.3.14. *Let $\nu \cdot \lambda \leq \pi$ and suppose that $\Psi(\nu, \lambda) < 1$ then $\mathbb{E}_\nu L_n \to \infty$ as $n \to \infty$.*

Proof. It follows from Lemma 4.3.12 that $\mathbb{P}_\nu(L_n > K \text{ infinitely often }) = 1 - \Psi(\nu, \lambda)$, for every constant $K > 0$. Let us thus pick a K (to be made explicit later), and let N be the (random) first index n such that $L_n > K$. (Here $N = \infty$ if no such n exists. Note $N < \infty$ with probability $1 - \Psi(\nu, \lambda) > 0$.) Let n_0 be such that $\mathbb{P}_\nu(N < n_0) > (1 - \Psi(\nu, \lambda))/2$. By conditioning on the value of N, we find that for $n \geq n_0$:

$$\begin{aligned}\mathbb{E}L_n &\geq \textstyle\sum_{m=0}^{n_0} \mathbb{E}(L_n | N = m)\mathbb{P}_\nu(N = m) \\ &= \textstyle\sum_{m=0}^{n_0} K \cdot 2^{n-m} \cdot \exp[-\sum_{i=m+1}^{n}(1+O((1/4)^i)) \cdot (\nu\lambda/\pi) \cdot \ln 2] \cdot \mathbb{P}_\nu(N = m) \\ &= \textstyle\sum_{m=0}^{n_0} K \cdot 2^{n-m} \cdot \exp[-(n-m) \cdot (\nu\lambda/\pi) \cdot \ln 2 + O(1)] \cdot \mathbb{P}_\nu(N = m) \\ &= \Omega\left(K \cdot \textstyle\sum_{m=0}^{n_0} 2^{(n-m)(1-\nu\lambda/\pi)}\mathbb{P}_\nu(N = m)\right) \\ &= \Omega\left(K \cdot \textstyle\sum_{m=0}^{n_0} \mathbb{P}_\nu(N = m)\right) \\ &= \Omega(K \cdot (1 - \Psi(\nu, \lambda))/2).\end{aligned}$$

Sending $K \to \infty$ proves the lemma. □

It follows immediately from Lemmas 4.3.13 and 4.3.14 that:

Corollary 4.3.15. *If $\nu\lambda = \pi$ then $\Psi(\nu, \lambda) = 1$.*

This last corollary of course also implies that $\Psi(\nu, \lambda) = 1$ for all $\nu \cdot \lambda \geq \pi$.

Lemma 4.3.16. *For every $\nu, \lambda > 0$ with $\nu \cdot \lambda < \pi$ there exists an $\eta_0 = \eta(\nu, \lambda)$ such that for every $0 < \eta < \eta_0$ we have*

$$\liminf_{n\to\infty} \mathbb{P}_\nu\big([0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_n\big) > 1/2.$$

Proof. Let μ_n denote the expected number of points $(r, \theta) \in \mathcal{A}_n$ that cover 0, and let $\tilde{\mu}_n$ denote expected number of points $(r, \theta) \in \mathcal{A}_n$ that cover *some* point of $[0, \eta \cdot 2^{-n})$. Then we have, similar to the proof of Lemma 4.3.5:

$$\begin{aligned} \tilde{\mu}_n &= \int_{r_{n-1}}^{r_n} \left(\eta \cdot 2^{-n} + 2\arccos\left(\tfrac{\cosh(r)-1}{\sinh(r)}\right)\right) \cdot g(r, \theta)\mathrm{d}r \\ &= \eta \int_{r_{n-1}}^{r_n} 2^{-n} \cdot (1 + O(e^{-r})) \cdot e^{r/2}\mathrm{d}r + \mu_n \\ &= (1 + o(1)) \cdot (\eta/2 + (\nu\lambda/\pi) \cdot \ln 2), \end{aligned} \tag{4.12}$$

reusing the computations (4.8) in the second line. Since $\nu\lambda < \pi$ we can choose $\eta > 0$ such that $\eta/2 + (\nu\lambda/\pi) \cdot \ln 2 < \ln 2$. In that case we have

$$\liminf_{n\to\infty} \mathbb{P}_\nu\big([0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_n\big) = \liminf_{n\to\infty} e^{-\tilde{\mu}_n} > 1/2,$$

as required. □

For the remainder of the section, we fix $\eta > 0$ such that the conclusion of the last lemma holds. Let us now consider the following random process. We start by dissecting $[0, 2\pi)$ into intevals $[0, \eta), [\eta, 2\eta), \ldots, [2\pi - \eta, 2\pi)$ of length η. (We assume without loss of generality that $\eta = \frac{2\pi}{k}$, for some k.) Each of these intervals "survives" if none of its points is covered by points of $\mathcal{P}$ of radius at most r_1. In each subsequent "generation", we split the surviving intervals in two, and these survive if none of their points are covered by a point of $\mathcal{P}$ of radius between r_{n-1} and r_n. This does produce a kind of branching process, but with the unfortunate property that the offspring of

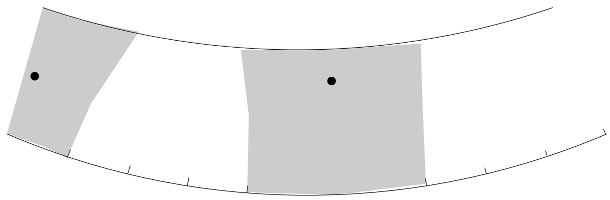

Figure 4.2: Depiction of the “particles” of the process.

different intervals in generation n are not always independent (e.g., if two intervals share an endpoint then their offspring are dependent, or more generally if they are close enough for a point of radius bigger than r_n to cover a point in each of the two intervals.)

To deal with this problem, we group the surviving intervals into “particles" consisting of (maximal) sequences of intervals each sharing an endpoint with the next. The type of a particle will be the number of intervals it consists of. See Figure 4.2 for a depiction.

Note that, in generation n, the gap between different particles is at least $2 \cdot \gamma(r_n)$. So no point of radius $> r_n$ can cover points in two different particles of generation n. This implies that the offspring distributions are independent.

Thus, we have defined a time-inhomogeneous multitype Galton-Watson process $Z_0^\lambda, Z_1^\lambda, \dots$ with countably many types. Again, we drop the superscript if it is clear from the context. Let $p_n(i; z_1, z_2, \dots)$ denote the probability that a particle of type i in generation n produces z_1 children of type 1, z_2 children of type 2 and so on. (Note that strictly speaking we would also need to introduce types for the case when $\mathcal{U}_n = [0, 2\pi)$ in which case there is one particle that “wraps around". This situation however does not occur as soon as there is at least one point with radius $\leq r_n$. So this is not a real issue. We leave it to the reader to check that the proofs below can be adapted to work also with this more proper but also more cumbersome definition of the process.)

Lemma 4.3.17. *For every $i, z_1, z_2, \dots$ the limits*

$$p(i; z_1, z_2, \dots) := \lim_{n\to\infty} p_n(i; z_1, z_2, \dots),$$

exist.

Proof. Let us fix $i, z_1, z_2, \dots$, and let E_n denote the event that $[0, i \cdot \eta \cdot 2^{-n})$ is split into a groups of intervals of length $2^{-(n+1)}$ in the required way by $\mathcal{A}_n$, i.e. among $[0, \eta \cdot 2^{-(n+1)}), \dots, [(2i-1) \cdot \eta \cdot 2^{-(n+1)})$ there are z_1 intervals such that none of their points are covered by $\mathcal{A}_n$ but some point in each of the neighbouring intervals were covered, and so on.

Let $A_n \subseteq \mathbb{H}$ denote the set of all points (r, θ) with $r_{n-1} < r \le r_n$ and $\theta \in (-10 \cdot 2^{-n}, (i \cdot \eta + 10) \cdot 2^{-n})$; and let $W_n := |\mathcal{P} \cap A_n|$ denote the number of points of $\mathcal{P}$ that fall inside A_n. By (4.7), for large enough n, whether or not E_n holds will only depend on the points of $\mathcal{P}$ that fall inside A_n. We have

$$p_n(i; z_1, z_2, \dots) = \mathbb{P}_\nu(E_n) = \sum_{t=0}^{\infty} \mathbb{P}_\nu(E_n | W_n = t) \mathbb{P}_\nu(W_n = t). \tag{4.13}$$

Let us observe that

$$\begin{aligned} \mathbb{E}W_n &= \int_{A_n} g(r,\theta) \mathrm{d}r\mathrm{d}\theta = 2^{-n} \cdot (i \cdot \eta + 20) \cdot (\nu\lambda/2\pi) \cdot 2(\cosh(r_n/2) - \cosh(r_{n-1}/2)) \\ &= 2^{-n} \cdot (i \cdot \eta + 20) \cdot (\nu\lambda/2\pi) \cdot (e^{r_n/2} + e^{-r_n/2} - e^{r_{n-1}/2} + e^{-r_{n-1}/2}) \\ &= (1 + o(1)) \cdot (i \cdot \eta + 20) \cdot (\nu\lambda/\pi). \end{aligned}$$

It follows also that W_n converges in distribution to a random variable distributed as $\mathrm{Po}\left((i \cdot \eta + 20)^{-1} \cdot (\nu\lambda/\pi)\right)$. Therefore, in the light of (4.13), in order to prove that $p_n(i; z_1, z_2, \dots)$ converges, it suffices to prove that the conditional probability $\mathbb{P}_\nu(E_n | W_n = t)$ converges for every fixed $t \in \mathbb{N}$. Let us thus fix a $t \in \mathbb{N}$.

Observe that if we condition on $W = t$ then $\mathcal{P} \cap A$ behaves like t i.i.d. random vectors $X_1 = (\rho_1, \theta_1), \dots, X_t = (\rho_t, \theta_t)$ with common probability density:

$$\tilde{g}(\rho,\theta) = \frac{g(\rho,\theta)}{\int_A g(r',\theta')\mathrm{d}r'\mathrm{d}\theta'} = (1+o_n(1)) \cdot (i\cdot\eta + 20)^{-1} \cdot e^{\rho/2},$$

where we used that $g(\rho,\theta) = (\nu/4\pi)\sinh(\rho/2) = (1+O(e^{-\rho}))\cdot(\nu/4)\cdot e^{\rho/2}$.

For notational convenience we write $I_j := [j\cdot\eta\cdot 2^{-(n+1)}, (j+1)\cdot\eta\cdot 2^{-(n+1)})$. For $0 \le j < 2i$ and $1 \le s \le t$ we set $F_n^{j,s} := \{(\rho_s,\theta_s) \text{ covers a point of } I_j\}$ and for $J \subseteq \{0,\dots,2i-1\}\times\{1,\dots,t\}$ we define

$$F_n^J := \left(\bigcap_{(j,s)\in J} F_n^{j,s}\right) \cap \left(\bigcap_{(j,s)\notin J} (F_n^{j,s})^c\right).$$

I.e., the event F_n^J prescribes precisely which of the t points covers which of the $2i$ intervals. Clearly there is some family of sets $\mathcal{J} \subseteq 2^{\{0,\dots,2i-1\}\times\{1,\dots,t\}}$ such that

$$\mathbb{P}_\nu(E_n | W_n = t) = \mathbb{P}_\nu\left(\bigcup_{J\in\mathcal{J}} F_n^J\right) = \sum_{J\in\mathcal{J}} \mathbb{P}_\nu(F_n^J).$$

It thus suffices to prove that the probabilities $\mathbb{P}_\nu(F_n^J)$ converge. Let us thus fix some $J \subseteq \{0,\dots,2i-1\}\times\{1,\dots,t\}$. Setting

$$\varphi_n^j(\rho,\theta) := \begin{cases} 1 & \text{if } \theta \in \big(j\cdot\eta\cdot 2^{-(n+1)} - \gamma(\rho), (j+1)\cdot\eta\cdot 2^{-(n+1)} + \gamma(\rho)\big); \\ 0 & \text{otherwise.} \end{cases},$$

and $\ell := -10\cdot 2^{-n}, u := (i\cdot\eta + 10)\cdot 2^{-n}$, we can write

$$
\begin{aligned}
&\mathbb{P}_\nu(F_n^J)\\
= &\int_\ell^u \int_{r_{n-1}}^{r_n} \dots \int_\ell^u \int_{r_{n-1}}^{r_n} \prod_{(j,s)\in J} \varphi_n^j(\rho_s,\theta_s) \cdot \prod_{(j,s)\notin J} (1-\varphi_n^j(\rho_s,\theta_s)) \cdot \prod_{s=1}^{t} \tilde{g}(\rho_s,\theta_s) \,\mathrm{d}\rho_1\mathrm{d}\theta_1 \dots \mathrm{d}\rho_t\mathrm{d}\theta_t\\
= &\int_{-10}^{i\eta+10} \int_0^{2\ln 2} \dots \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \prod_{(j,s)\in J} \varphi_n^j(r_{n-1}+x_s, 2^{-n}\cdot\vartheta_s)\cdot\\
&\qquad \prod_{(j,s)\notin J} (1-\varphi_n^j(r_{n-1}+x_s, 2^{-n}\vartheta_s)) \cdot \prod_{s=1}^{t} \tilde{g}(r_{n-1}+x_s, 2^{-n}\vartheta_s) \cdot 2^{-t\cdot n} \,\mathrm{d}x_1\mathrm{d}\vartheta_1 \dots \mathrm{d}x_t\mathrm{d}\vartheta_t\\
= &\int_{-10}^{i\eta+10} \int_0^{2\ln 2} \dots \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \prod_{(j,s)\in J} \varphi_n^j(r_{n-1}+x_s, 2^{-n}\cdot\vartheta_s)\cdot \prod_{(j,s)\notin J} (1-\varphi_n^j(r_{n-1}+x_s, 2^{-n}\vartheta_s))\cdot\\
&\qquad (1+o_n(1))\cdot(i\cdot\eta+20)^{-t}\cdot e^{\sum_{s=1}^t (r_{n-1}+x_i)/2}\cdot 2^{-t\cdot n} \,\mathrm{d}x_1\mathrm{d}\vartheta_1 \dots \mathrm{d}x_t\mathrm{d}\vartheta_t\\
= &\int_{-10}^{i\eta+10} \int_0^{2\ln 2} \dots \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \prod_{(j,s)\in J} \varphi_n^j(r_{n-1}+x_s, 2^{-n}\cdot\vartheta_s)\cdot \prod_{(j,s)\notin J} (1-\varphi_n^j(r_{n-1}+x_s, 2^{-n}\vartheta_s))\cdot\\
&\qquad (1+o_n(1))\cdot(i\cdot\eta+20)^{-t}\cdot 2^{-t}\cdot e^{(x_1+\dots+x_t)/2} \,\mathrm{d}x_1\mathrm{d}\vartheta_1 \dots \mathrm{d}x_t\mathrm{d}\vartheta_t,
\end{aligned}
$$

applying the substitutions $r_s = r_{n-1} + x_s, \theta_s = 2^{-n}\vartheta_s$ in the second line. Let us now define, for $0 \le x \le 2\ln 2$ and $-10 \le \vartheta \le i\cdot\eta + 10$:

$$
\psi^j(x,\vartheta) := \begin{cases} 1 & \text{if } \vartheta \in (j\cdot\eta/2 - e^{-x/2}, (j+1)\cdot\eta/2 + e^{-x/2}),\\ 0 & \text{otherwise.}\end{cases}
$$

It follows from (4.7) that

$$
\lim_{n\to\infty} \varphi_n^j(r_{n-1}+x, 2^{-n}\vartheta) = \psi^j(x,\vartheta) \qquad \text{almost everywhere.}
$$

(Recall that *almost everywhere* means “for all (x,ϑ) except for a set of Lebesgue measure zero".) Using the dominated convergence theorem we can now conclude that

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{P}_\nu(F_n^J) \quad = \quad &(2i\cdot\eta+40)^{-t} \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \dots \int_{-10}^{i\eta+10} \int_0^{2\ln 2} \prod_{(j,s)\in J} \psi^j(x_s,\vartheta_s)\cdot\\
&\prod_{(j,s)\notin J} (1-\psi^j(x_s,\vartheta_s))\cdot e^{(x_1+\dots+x_t)/2} \,\mathrm{d}x_1\mathrm{d}\vartheta_1 \dots \mathrm{d}x_t\mathrm{d}\vartheta_t.
\end{aligned}
$$

The lemma follows. ◻

Lemma 4.3.18. *The limits $p(i; z_1, z_2, \dots)$ from Lemma 4.3.17 satisfy the conditions of Lemma 4.2.4.*

Proof. Let us first note that the expression $\sum_j m_{ij}$ simply counts the expected (total) number of intervals of length $\eta \cdot 2^{-(n+1)}$ in the offspring of a type i particle. An uncovered interval I of length $\eta \cdot 2^{-n}$ in generation n will get split into two uncovered intervals of length $\eta \cdot 2^{-(n+1)}$ in generation $n+1$ if no point of $\mathcal{A}_n$ covers a point of I. It thus follows immediately from the choice of η (cf. Lemma 4.3.16) that

$$\sum_j m_{ij} \geq \liminf_{n\to\infty} 2i \cdot \mathbb{P}_\nu([0, \eta \cdot 2^{-n}) \in \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \in \mathcal{U}_n) = c \cdot i,$$

where $c := 2 \cdot \liminf_{n\to\infty} \mathbb{P}_\nu([0, \eta \cdot 2^{-n}) \in \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \in \mathcal{U}_n) > 1$. This verifies the first condition of Lemma 4.2.4.

The third condition follows immediately from the fact that the total length of the offspring of a particle is never more than the length of the particle.

To see that the second condition holds, it suffices to show that the probability that a particle of type i gives birth to at least one particle of type j is bounded away from zero whenever $j \leq i$. To this end, let $\mu_n^{(i)}$ denote the expected number of points $(r, \theta) \in \mathcal{A}_n$ that cover some angle of $[0, j \cdot \eta \cdot 2^{-(n+1)})$. By an almost verbatim repeat of the computations (4.12) we have

$$\begin{aligned}
\mu_n^{(i)} &= \int_{r_{n-1}}^{r_n} \left(j \cdot \eta \cdot 2^{-(n+1)} + 2\gamma(r)\right) \cdot g(r, \theta) \mathrm{d}r \\
&= (1 + o(1)) \cdot (j \cdot \eta/4 + (\nu\lambda/\pi) \cdot \ln 2),
\end{aligned}$$

Let E denote the event that that $\mathcal{A}_n$ covers no angle of $[0, j \cdot \eta \cdot 2^{-(n+1)})$ but some angle of $[0, (j+1) \cdot \eta \cdot 2^{-(n+1)})$. Since the probability that a particle of type j is born among the offspring of a type i particle is at least the probability that E holds, we have that

$$
\begin{aligned}
m_{ij} &\geq \mathbb{P}_\nu(E) \\
&= \lim_{n\to\infty} \mathbb{P}(\mathrm{Po}(\mu_n^{(j)}) = 0) \cdot \mathbb{P}(\mathrm{Po}(\mu_n^{(j+1)} - \mu_n^{(j)}) > 0) \\
&= \lim_{n\to\infty} (\mu_n^{(j+1)} - \mu_n^{(j)}) \cdot e^{-\mu_n^{(j+1)}} \\
&= (\eta/4) \cdot e^{-(j+1)\cdot\eta/4 + (\nu\lambda/\pi)\cdot\ln 2} \\
&> 0.
\end{aligned}
$$

It remains to check that the fourth condition holds. To this end, observe that if we cut an interval of length $i \cdot \eta \cdot 2^{-n}$ into four equal parts, then if $\mathcal{A}_n$ covers at least one point in each part, then the offspring of the original type-I particle will consist of particles of types $\leq i$. Hence, we have:

$$
\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1}+z_{i+2}+\dots > 0}} p(i; z_1, z_2, \dots) \leq 1 - \liminf_{n\to\infty} (1 - e^{-\mu_n^{(\lfloor i/4 \rfloor)}})^4 = 1 - \left(1 - e^{-(\lfloor i/4 \rfloor \cdot \eta/4 + (\nu\lambda/\pi)\cdot \ln 2)}\right)^4.
$$

It is clear that if we send $i \to \infty$ then this last expression approaches zero. This proves that the fourth condition holds, and finishes the proof of the lemma. □

Invoking Lemma 4.2.5, we have the following immediate corollary.

Corollary 4.3.19. *If* $\nu \cdot \lambda < \pi$ *then* $\liminf_{n\to\infty} \mathbb{P}_\nu(Z \text{ explodes } | Z_n = e_1) > 0$.

We are now also able to deduce:

Lemma 4.3.20. *If* $\nu\lambda < \pi$ *then* $\Psi(\nu, \lambda) < 1$.

Proof. Observe that the event that Z explodes is contained in the event that $\mathcal{C}(\lambda)$ does not occur. By Corollary 4.3.19 we can pick $n \in \mathbb{N}$ such that $\mathbb{P}_\nu(Z \text{ explodes } | Z_n = e_1) > 0$. Let E denote the event that $\mathcal{B}_n = \emptyset$, i.e. no point of $\mathcal{P}_\nu$ has radius $\leq r_n$. Then we have that

$$
\mathbb{P}_\nu(E) = \exp[-(\nu/2) \cdot (\cosh(r_n/2) - 1)] > 0.
$$

We have

$$\begin{aligned}1 - \Psi(\nu, \lambda) &= \mathbb{P}_\nu(\text{not } \mathcal{C}(\lambda)) \\ &\geq \mathbb{P}_\nu(E) \cdot \mathbb{P}_\nu(Z \text{ explodes}|E) \\ &\geq \mathbb{P}_\nu(E) \cdot \mathbb{P}_\nu(Z \text{ explodes}|Z_n = e_1) \\ &> 0,\end{aligned}$$

where the penultimate inequality holds by obvious monotonicity. □

Lemma 4.3.21. *For every* $\lambda > 0$ *it holds that* $\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) = 0$

Proof. The proof is very similar to the previous lemma. Let us first observe that for every fixed n the conditional probability $\mathbb{P}_\nu(Z \text{ explodes } |Z_n = e_1)$ is nonincreasing in ν. (This can for instance be seen by noting that a Poisson process with intensity function $g_{\nu+\delta}(r, \theta)$ is the superposition of one with density function g_ν and one with density function g_δ.) Hence we can find an $n_0 \in \mathbb{N}$ and $c > 0$ such that $\mathbb{P}_\nu(Z \text{ explodes}|Z_n = e_1) \geq c$ for all $n \geq n_0$ and all $0 < \nu < 1$. Now note that for every $K > 0$, there exists an n such that among $[0, \eta \cdot 2^{-n}), \ldots, [2\pi - \eta \cdot 2^{-n}, 2\pi)$ there are at least K intervals that are separated by pairwise distance of at least $2\gamma(r_n)$. Fix such an n, and let E denote the event that no point fell inside $\mathcal{B}_n$.

Then we have

$$1-\Psi(\nu, \lambda) \geq \mathbb{P}_\nu(E)\cdot\big(1 - \mathbb{P}_\nu(Z \text{ dies out } |Z_n = e_1)^K\big) \geq e^{-(\nu/2)\cdot(\cosh(r_n/2)-1)}\cdot(1-(1-c)^K).$$

Let $\varepsilon > 0$ be arbitrary. By choosing K sufficiently large, we can ensure that $(1-c)^K < \varepsilon$. It follows that

$$\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) \leq 1 - \lim_{\nu \downarrow 0} e^{-(\nu/2)\cdot(\cosh(r_n/2)-1)} \cdot (1 - \varepsilon) = \varepsilon.$$

Sending ε to zero finishes the proof. □

Let $L_n^\eta = L_n^\eta(\lambda)$ denote the total length of all components of $L_n(\lambda)$ that have length at least $\eta \cdot 2^{-n}$. As usual, when λ is clear from the context we omit it. A similar proof to that of the previous lemma also gives the following.

Lemma 4.3.22. *If $\nu\lambda < \pi$ and $K > 0$ arbitrary then $\mathbb{P}_\nu(L_n^\eta > K$ almost always $) = 1 - \Psi(\nu, \lambda)$.*

Proof. Observe that if $L_n^\eta > K$ almost always, then $\mathcal{C}(\lambda)$ certainly does not occur. This shows that

$$\mathbb{P}_\nu(L_n^\eta > K \text{ almost always }) \leq 1 - \Psi(\nu, \lambda).$$

Also observe that if Z explodes, then we also have that $L_n^\eta > K$ almost always.

Now let $\varepsilon > 0$ be arbitrary and let us fix a $K' = K'(\varepsilon, K)$, to be made precise later. By Lemma 4.3.12, we have that $\mathbb{P}_\nu(L_n^{\text{long}} > K'$ infinitely often $) = 1 - \Psi(\nu, \lambda)$. As in the proof of the previous lemma, we can pick $n_0, c > 0$ such that $\mathbb{P}_\nu(Z$ explodes $|Z_n = e_1) \geq c$ for all $n \geq n_0$.

Observe that if $L^{\text{long}} > K'$ then we can find a family of at least

$$M := \left\lceil \frac{K' \cdot 2^{-n}}{\eta \cdot 2^{-n} + 2\gamma(r_n)} \right\rceil,$$

intervals of length $\eta \cdot 2^{-n}$ in $\mathcal{U}_n$ that are separated by pairwise distance $2\gamma(r_n)$. By (4.7), we have that $M > K'/10$ for sufficiently large n.

Now consider the following setup. We let N denote the (random) first integer after n_0 for which $L_n^{\text{long}} > K'$, where $N = \infty$ if there is no such N. Note that the event $N = n$ is independent of $\mathcal{P} \setminus B_{\mathbb{H}}(O; r_n)$. This shows that

$$\begin{aligned}
\mathbb{P}_\nu(Z \text{ explodes}) &\geq \textstyle\sum_{n=n_0}^\infty \mathbb{P}_\nu(N=n)\cdot\big(1-\mathbb{P}_\nu(Z \text{ dies out}|Z_n=e_1)^M\big)\\
&\geq \textstyle\sum_{n=n_0}^\infty \mathbb{P}_\nu(N=n)\cdot\big(1-(1-c)^M\big)\\
&\geq \textstyle\sum_{n=n_0}^\infty \mathbb{P}_\nu(N=n)\cdot(1-\varepsilon)\\
&= \mathbb{P}_\nu(N<\infty)\cdot(1-\varepsilon)\\
&\geq \mathbb{P}_\nu(L_n^{\text{long}} > K' \text{ infinitely often })\cdot(1-\varepsilon)\\
&= (1-f(\nu))\cdot(1-\varepsilon).
\end{aligned}$$

Sending ε to zero gives the lemma. □

Let us define

$$\Psi_n(\nu,\lambda) := \mathbb{P}_\nu(\mathcal{C}_{r_n}(\lambda)).$$

In other words, Ψ_n is the probability that $\mathcal{B}_n$ is a cover.

Lemma 4.3.23. *Let $s>0$ be fixed, but arbitrary. Let F be any event that depends only on $\mathcal{P}_\nu \cap B_{\mathbb{H}}(0,s)$ (i.e. F of radius less than s), and set $\varphi(\nu) := \mathbb{P}_\nu(F)$. Then φ is a continuous function of ν.*

Proof. Let Y denote the number of points of $\mathcal{P}$ with radius at most s. Then Y is Poisson-distributed with mean $\mathbb{E}Y = \nu\cdot(\cosh(s/2)-1)$. Let us remark that

$$a_t := \mathbb{P}_\nu(F|Y=t),$$

is independent of ν. (To see this, note that if we condition on $Y=t$ then the points of $\mathcal{P}$ with radius $\leq s$ behave like an i.i.d. sample $X_1,\dots,X_t$ with common density function

$$h(r,\theta) = \frac{g(r,\theta)}{\int_0^{2\pi}\int_0^s g(t,\beta)\mathrm{d}t\mathrm{d}\beta} = \frac{\sinh(r/2)}{2\pi\cdot(\cosh(s/2)-1)}.$$

The function h is clearly independent of ν.) We clearly have

$$\varphi(\nu) = \sum_{t=0}^{\infty} a_t \cdot \mathbb{P}_\nu(Y = t).$$

Let us now fix an arbitrary $\varepsilon > 0$. Set $K := 1000 \cdot \mathbb{E}_\nu Y/\varepsilon$. By Markov's inequality we have $\mathbb{P}_\mu(Y \geq K) \leq \mathbb{E}_\mu Y/K \leq \varepsilon/2$, for all $\mu < 500\nu$. Hence, for all $\mu < 500\nu$ we have

$$\left|\varphi(\mu) - \sum_{t=0}^{K} a_t \cdot p_t(\mu)\right| < \varepsilon/2,$$

where $p_t(\mu) := \mathbb{P}_\mu(Y = t) = (\mu \cdot (\cosh(s/2) - 1))^t \cdot e^{-\mu\cdot(\cosh(s/2)-1)}/t!$. Now observe that p_t is a continuous function of μ for every (fixed) t. It follows that there is a $\delta > 0$ such that if $|\mu - \nu| < \delta$ then $|p_t(\mu) - p_t(\nu)| < \varepsilon/2(K+1)$ for all $0 \leq t < K$. Hence we also have that $|\varphi(\mu) - \varphi(\nu)| < \varepsilon$ whenever $|\mu - \nu| < \min(\delta, 499\nu)$. This proves that φ is continuous as claimed. $\square$

Corollary 4.3.24. *For every $n \in \mathbb{N}$, the function Ψ_n is continuous in its first parameter, ν.*

Lemma 4.3.25. *For every $n \in \mathbb{N}$, the function Ψ_n is continuous in its second parameter, λ.*

Proof. Let us fix ν. Let us take $\lambda_1 < \lambda_2$ and let us write $\gamma_i(r) = \lambda_i \arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)$ for $i = 1, 2$. Note that $\Psi_n(\nu, \lambda_2) - \Psi_n(\nu, \lambda_1)$ is precisely the probability of the event E that $\bigcup_{(r,\theta)\in\mathcal{B}_n}(\theta - \gamma_2(r), \theta + \gamma_2(r))$ covers all angles, but some angle is not covered by $\bigcup_{(r,\theta)\in\mathcal{B}_n}(\theta - \gamma_1(r), \theta + \gamma_1(r))$.

Next, let us observe that if E holds then there must exist two points $(r, \theta), (s, \vartheta) \in \mathcal{B}_n$ such that

$$\gamma_1(r) + \gamma_1(s) < |\theta - \vartheta|_{2\pi} < \gamma_2(r) + \gamma_2(s). \tag{4.14}$$

(Consider some component I of $\mathcal{U}_n$ under λ_1. The leftmost endpoint of this interval is the rightmost endpoint of $(\theta - \gamma_1(r), \theta + \gamma_2(r))$ for some $(r, \theta) \in \mathcal{B}_n$. Since $\mathcal{C}(\lambda)$ occurs at λ_2, it must be the case that $\theta + \gamma_2(r)$ is inside some interval $(\vartheta - \gamma_2(s), \vartheta + \gamma_2(s))$.) From this it follows that

$$\mathbb{P}_\nu(E) \leq (\mathbb{E}_\nu|\mathcal{B}_n|)^2 \cdot \mathbb{P}_\nu\left(|\theta - \vartheta|_{2\pi} \in (\gamma_1(r) + \gamma_1(s), \gamma_2(r) + \gamma_2(s))\right),$$

where the points (r, θ) and (s, ϑ) are chosen i.i.d. according to the distribution with density $g / \int_{B_{\mathbb{H}}(O,R)} \int_0^{2\pi} g$. (We used Palm Theory for counting the number of pairs with this property.)

Now note that the length of the interval $(\lambda_1(r) + \lambda_1(s), \lambda_2(r) + \lambda_2(s))$ is at most $2(\lambda_2 - \lambda_1) \lim_{x \downarrow 0} \arccos\left(\frac{\cosh(x)-1}{\sinh(x)}\right) = (\lambda_2 - \lambda_1) \cdot \pi$. It follows that

$$\mathbb{P}_\nu(E) \leq (\mathbb{E}_\nu|\mathcal{B}_n|)^2 \cdot \frac{\lambda_2 - \lambda_1}{2}.$$

Thus, by choosing λ_1, λ_2 such that $\lambda_2 - \lambda_1 < 2\varepsilon / (\mathbb{E}_\nu|\mathcal{B}_n|)^2$, we can ensure that $|\Psi_n(\nu, \lambda_2) - \Psi_n(\nu, \lambda_1)| \leq \mathbb{P}_\nu(E) < \varepsilon$. This proves that Ψ_n is indeed continuous in λ. □

Next, we define, for every $\eta, K > 0$ and $n \in \mathbb{N}$:

$$\Phi_{n,\eta,K}(\nu, \lambda) := \mathbb{P}_\nu(L_n^\eta > 0).$$

By an application of Lemma 4.3.23, we find that:

Corollary 4.3.26. *$\Phi_{n,\eta,K}$ is continuous in its first parameter, ν. (For every $\eta, K > 0$ and $n \in \mathbb{N}$.)*

Lemma 4.3.27. *$\Phi_{n,\eta,K}$ is continuous in its second parameter, λ. (For every $\eta, K > 0$ and $n \in \mathbb{N}$.)*

Proof. To begin, we fix $\nu, \lambda, \eta, K > 0$ and $n \in \mathbb{N}$. Observe that there exists some $\delta > 0$ such that

$$\mathbb{P}_\nu(L_n^\eta \geq K + \delta) \geq \Phi_{n,\eta,K}(\nu, \lambda_1) - \varepsilon/3. \tag{4.15}$$

Similarly, we may assume that δ is small enough so that

$$\mathbb{P}_\nu(\mathcal{U}_n \text{ has a component of length} \in [\eta 2^{-n} - \delta, \eta 2^{-n} + \delta]) < \varepsilon/3. \tag{4.16}$$

(Arguing as in the proof of Lemma 4.3.25, but now considering pair of points whose distance is close to $\gamma(r) + \gamma(s) + \eta 2^{-n}$.)

Finally let us pick some $\lambda' \neq \lambda$, and let the sum $\sum_{(r,\theta)\in\mathcal{B}_n} 2|\lambda' - \lambda| \arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)$ be denoted by X. (I.e., X is the sum over all points in $\mathcal{B}_n$ of the difference in the covered length under the two choices of the parameter λ.) Using Markov's inequality, we have that

$$\mathbb{P}_\nu(X > \delta) \leq \frac{\mathbb{E}_\nu X}{\delta} \leq \mathbb{E}_\nu|\mathcal{B}_n| \cdot \pi \cdot |\lambda' - \lambda| < \varepsilon/3, \tag{4.17}$$

we the last inequality holds for $|\lambda' - \lambda|$ sufficiently small.

Observe that if $L_n^\eta \geq K + \delta$ with respect to λ, there are no components in $\mathcal{U}_n$ of length $\in [\eta 2^{-n} - \delta, \eta 2^{-n} + \delta]$, and $X \leq \delta$, then $L_n^\eta > K$ with respect to λ. Thus, combining (4.15), (4.16) and (4.17), we have proved the lemma. □

Lemma 4.3.28. Ψ *is continuous.*

Proof. Let $\nu, \lambda > 0$ be abtritrary. We first assume that $\nu\lambda \geq \pi$. In this case $\Psi(\nu, \lambda) = 1$ by Corollary 4.3.15. Note that, since $\mathcal{C}(\lambda) = \bigcup_n \mathcal{C}_{r_n}(\lambda)$, there exists an n such that $\Psi_n(\nu, \lambda) \geq 1 - \varepsilon/2$. Since Ψ_n is continuous, there is a $\delta > 0$ such that

$$\Psi(\nu', \lambda') \geq \Psi_n(\nu', \lambda') \geq \Psi_n(\nu, \lambda) - \varepsilon/2 \geq 1 - \varepsilon,$$

for all $\nu' \in (\nu - \delta, \nu + \delta)$ and $\lambda' \in (\lambda - \delta, \lambda + \delta)$. This shows Ψ is continuous at ν, λ.

Let us then assume that $\nu\lambda < \pi$. Let us pick $\nu' > \nu, \lambda' > \lambda$ such that still $\nu'\lambda' < \pi$; and let $n_0 \in \mathbb{N}, c > 0$ be such that

$$\mathbb{P}_{\nu'}(Z^{\lambda'} \text{ explodes } | Z_n^{\lambda'} = e_1) \geq c,$$

for all $n \geq n_0$. Note that, by obvious monotonicity, this inequality also holds for all $\nu'' < \nu', \lambda'' < \lambda'$ (here we keep η, used in the definition of the process Z, fixed).

Let $\varepsilon > 0$ be arbitrary and let $K = K(\varepsilon)$ be fixed to be made precise later. Since $\Psi(\nu, \lambda) = \lim_{n\to\infty} \Psi_n(\nu, \lambda)$, we can find an n_1 such that $|\Psi_n(\nu, \lambda) - \Psi(\nu, \lambda)| < \varepsilon/2$ for all $n \geq n_1$. Similarly, since

$$1 - \Psi(\nu, \lambda) = \mathbb{P}_\nu(L_n^\eta(\lambda) > K \text{ almost always }) = \lim_{n\to\infty} \mathbb{P}_\nu(L_m^\eta(\lambda) > K \text{ for all } m \geq n \text{ }),$$

we can fix an n_2 such that $\Phi_{n,\eta,K}(\nu, \lambda) = \mathbb{P}_\nu(L_n^\eta > K) \geq 1 - \Psi(\nu, \lambda) - \varepsilon/2$ for all $n \geq n_2$.

Let us now fix $n := \max\{n_0, n_1, n_2\}$ and put $\varphi(\nu) := \mathbb{P}_\nu(L_n = 0), \psi(\nu) = \mathbb{P}_\nu(Z_n > K)$.

Since both Ψ_n and $\Phi_{n,\eta,K}$ are continuous, we can pick a $\delta > 0$ such that $|\Psi_n(\nu'', \lambda'') - \Psi_n(\nu, \lambda)| < \varepsilon/2$ and $|\Phi_{n,\eta,K}(\nu'', \lambda'') - \Phi_{n,\eta,K}(\nu, \lambda)| < \varepsilon/2$ for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$. We assume without loss of generality that $\delta < \min(\lambda' - \lambda, \nu' - \nu)$.

Now note that if $L_n^\eta(\lambda) > K$ then there are at least

$$M := \left\lceil \frac{K \cdot \eta \cdot 2^{-n}}{\eta \cdot 2^{-n} + 2\gamma(r_n)} \right\rceil = \Omega(K),$$

intervals of length at least $\eta \cdot 2^{-n}$ that are contained in $\mathcal{U}_n$ and that are separated by pairwise distance $2\gamma(r_n)$. It follows that, for all $\nu'' \in (\nu-\delta, \nu+\delta)$ and $\lambda'' \in (\lambda-\delta, \lambda+\delta)$, we have

$$\begin{aligned}
& \mathbb{P}_{\nu''}(L_m^\eta(\lambda'') > K \text{ almost always} | L_n^\eta(\lambda'') > K) \\
\geq\ & 1 - \mathbb{P}_{\nu''}(Z(\lambda'') \text{ dies out } | Z_n(\lambda'') = e_1)^M \\
\geq\ & 1 - (1-c)^M \\
\geq\ & 1 - \varepsilon/2,
\end{aligned}$$

where the last inequality holds provided we chose K sufficiently large (which we can assume without loss of generality). We thus get that

$$\begin{aligned} 1 - \Psi(\nu'', \lambda'') &= \mathbb{P}_{\nu'',\lambda''}(\text{not } \mathcal{C}(\lambda)) \\ &\geq \mathbb{P}_{\nu'',\lambda''}(Z \text{ explodes } | L_n^\eta > K)\Phi_{n,\eta,K}(\nu'', \lambda'') \\ &\geq (1-\varepsilon/2) \cdot (1 - \Psi(\nu, \lambda) - \varepsilon/2) \\ &\geq 1 - \Psi(\nu, \lambda) - \varepsilon, \end{aligned}$$

for all $\nu'' \in (\nu-\delta, \nu+\delta)$ and $\lambda'' \in (\lambda-\delta, \lambda+\delta)$. In other words, $\Psi(\nu'', \lambda'') \leq \Psi(\nu, \lambda) + \varepsilon$ for all $\nu'' \in (\nu-\delta, \nu+\delta)$ and $\lambda'' \in (\lambda-\delta, \lambda+\delta)$. On the other hand we have

$$\Psi(\nu'', \lambda'') \geq \Psi_n(\nu'', \lambda'') \geq \Psi(\nu, \lambda) - \varepsilon,$$

for all $\nu'' \in (\nu-\delta, \nu+\delta)$ and $\lambda'' \in (\lambda-\delta, \lambda+\delta)$, by choice of n and δ. We have seen that Ψ is continuous at (ν, λ) as required. □

We have already proved Theorem 4.3.1, but for completeness we collect our findings from this Section in an explicit proof.

Proof of Theorem 4.3.1: That Ψ is continuous was just established in the previous lemma. That $\Psi(\nu, \lambda) = 1$ when $\nu\lambda \geq \pi$ was established in Corollary 4.3.15. That Ψ is strictly increasing at every point $(\nu, \lambda$ with $\nu\lambda < \pi$ follows from Corollary 4.3.4 together with Lemma 4.3.20. That $\lim_{\nu\downarrow 0} \Psi(\nu, \lambda) = 0$ was established in Lemma 4.3.21. ■

4.4 The proof of part 3 of Theorem 1.6.3

Here, we finally prove the remaining part of Theorem 1.6.3, making use of Theorem 4.3.1.

Lemma 4.4.1. *Let $\mathcal{P} = \mathcal{P}_\nu$ be as defined earlier. For every $\varepsilon > 0$ there is a coupling such that $\mathcal{P}_{\nu-\varepsilon} \cap B_{\mathbb{H}}(O; R) \subseteq V_N \subseteq \mathcal{P}_{\nu+\varepsilon} \cap B_{\mathbb{H}}(O; R)$ w.h.p. as $N \to \infty$.*

Proof. Let $X_1, X_2, \ldots$ be an infinite supply of i.i.d. points distributed according to (1.1). Then we can set $V = \{X_1, \ldots, X_N\}$. Now let $Z_1 \stackrel{d}{=} \text{Po}((1-\delta)N)$, $Z_2 \stackrel{d}{=} \text{Po}((1+$

$\delta)N)$ and set $V_i := \{X_1, \dots, X_{Z_i}\}$ for $i = 1, 2$. It follows from the Chebyschev inequality that

$$\mathbb{P}_\nu(Z_1 \le N \le Z_2) = 1 - o(1).$$

Put differently, this proves that a.a.s. $V_1 \subseteq V_N \subseteq V_2$.

Now observe that V_1 is a Poisson process with intensity function:

$$\begin{aligned} h_1(r,\theta) &= (1-\delta)N \cdot (1/2\pi) \cdot \tfrac{(1/2)\cdot\sinh(r/2)}{\cosh(R/2)-1} \cdot 1_{\{r \le R\}} \\ &= (1-\delta)\nu e^{R/2} \cdot (1/2\pi) \cdot \tfrac{(1/2)\cdot\sinh(r/2)}{\cosh(R/2)-1} \cdot 1_{\{r \le R\}} \\ &= (1-\delta+o(1)) \cdot (\nu/4\pi) \cdot \sinh(r/2) \cdot 1_{\{r \le R\}}. \end{aligned}$$

So, provided we chose $\delta = \delta(\varepsilon)$ sufficiently small, we have $h_1(r,\theta) \ge g_{\nu-\varepsilon}(r,\theta)1_{\{r\le R\}}$ for all r, θ if N is sufficiently large (where g is the density of $\mathcal{P}$ defined in (4.2)). Similarly the density h_2 of V_2 satisfies $h_2 \le g_{\nu+\varepsilon}1_{\{r \le R\}}$ for N sufficiently large. The statement follows. □

Lemma 4.4.2. *For every* $\nu > 0$ *we have* $\liminf_{N\to\infty} \mathbb{P}(G(N; 1/2, \nu)$ *is connected* $) \ge \Psi(\nu, 1)$.

Proof. Let us pick a $\delta > 0$ such that $\Psi(\nu-\delta, 1-\delta) > \Psi(\nu, 1) - \varepsilon/3$. For convenience we write $\mu := \nu - \delta, \lambda := 1 - \delta$. Next, let us pick $s > 0$ such that $\mathbb{P}_\mu(\mathcal{C}_s(\lambda)) \ge \Psi(\mu, \lambda) - \varepsilon/3$. This is possible as $\mathcal{C}_s \subseteq \mathcal{C}_{s'}$ for $s < s'$, so $\mathbb{P}_\mu(\mathcal{C}_s(\lambda))$ is nondecreasing in s with limit $\mathbb{P}_\mu(\mathcal{C}(\lambda)) = \Psi(\mu, \lambda)$. Let us consider the coupling from the previous lemma. Taking N sufficiently large, we can assume that the probability that it fails is at most $\varepsilon/3$ and that $s < R/2$. (Recall that $R = R(N)$ depends on and is growing with N.)

We claim that, if $\mathcal{C}_s(\lambda)$ occurs with respect to μ, and the coupling succeeds (i.e. $\mathcal{P}_\mu \cap B_{\mathbb{H}}(O, R) \subseteq V_N$), then the graph $G(N; 1/2, \nu)$ will be connected. To see this suppose that $\mathcal{C}_s(\lambda)$ occurs with respect to μ, and pick an arbitrary point $X_i = (\rho_i, \theta_i) \in V_N$. There is some point $X_j = (\rho_j, \theta_j) \in V_N$ with $\rho_j \le s$ such that $|\rho_i - \rho_j|_{2\pi} < \gamma(\rho_j) = \lambda \cdot \arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right)$.

We claim that X_i and X_j must have distance less than R. To see this, note first that we are done when $\rho_i \leq R/2$ (using as $\rho_j \leq s < R/2$ and the triangle inequality). By the hyperbolic law of cosines (Fact 2.1.3) we have that the distance between X_i and X_j is less than R if and only if

$$|\theta_i - \theta_j|_{2\pi} < \arccos\left(\frac{\cosh(\rho_i)\cosh(\rho_j) - \cosh(R)}{\sinh(\rho_i)\sinh(\rho_j)}\right).$$

Now notice that

$$\begin{aligned}\arccos\left(\tfrac{\cosh(\rho_i)\cosh(\rho_j)-\cosh(R)}{\sinh(\rho_i)\sinh(\rho_j)}\right) &\leq \arccos\left(\tfrac{\cosh(\rho_i)\cosh(\rho_j)-\cosh(\rho_i)}{\sinh(\rho_i)\sinh(\rho_j)}\right)\\ &= \arccos\left(\tfrac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right).\end{aligned}$$

Recall that $(\cosh(r)-1)/\sinh(r) = 1-2e^{-r}+o(e^{-r})$ and note that $\cosh(\rho_i)/\sinh(\rho_i) = 1 + O(e^{-2\rho_i}) = 1 + O(e^{-R})$. Using Taylor's expansion $\arccos(x+y) = \arccos(x) - y/\sqrt{1-x^2} + O(xy^2/(1-x^2)^{3/2})$, we see that

$$\begin{aligned}\arccos\left(\tfrac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) &= \arccos\left(\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)} + O(e^{-R})\right)\\ &= \arccos\left(\tfrac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) + O(e^{\rho_j-R}).\end{aligned}$$

Using equations (4.6) and (4.7), we find that

$$\arccos\left(\frac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) = (1+o(1))\cdot\arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right).$$

Since $|\theta_i - \theta_j|_{2\pi} \leq \gamma(\rho_j) = (1-\delta)\cdot\arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right)$, we do find that X_i, X_j have distance at most R (for N sufficiently large).

This shows that, provided $\mathcal{C}_s(\lambda)$ occurs with respect to μ and the coupling succeeds (i.e. $\mathcal{P}_\mu \cap B_{\mathbb{H}}(O,R) \subseteq V_N$), then every vertex of $G(N;1/2,\nu)$ will be at distance less that R from some vertex of radius $< R/2$. So the graph will have diameter at most three, and in particular it will be connected. That is, we have shown

$$\liminf_{N\to\infty} \mathbb{P}(G(N;1/2,\nu) \text{ is connected }) \geq \mathbb{P}_\mu(\mathcal{C}_s(\lambda)) - \mathbb{P}(\text{the coupling fails}) \geq \Psi(\nu,1) - \varepsilon.$$

Sending ε to zero proves the lemma. □

Lemma 4.4.3. *For every $\nu > 0$ we have* $\limsup_{N\to\infty} \mathbb{P}(G(N;1/2,\nu)$ *is connected* $) \leq \Psi(\nu, 1)$.

Proof. If $\nu > \pi$ then there is nothing to prove as $\Psi(\nu,1) = 1$. Let us thus suppose that $\nu < \pi$ so that $\Psi(\nu, 1) < 1$. Reformulating, it suffices to show that

$$\liminf_{N\to\infty} \mathbb{P}(G(N;1/2,\nu) \text{ is NOT connected }) \geq 1 - \Psi(\nu, 1).$$

Pick a $\delta > 0$ such that $\Psi(\nu+\delta, 1+\delta) \leq \Psi(\nu,1) + \varepsilon/2$ and write $\mu := \nu+\delta, \lambda := 1+\delta$. Let K be large but fixed, to be made more precise later; and let $\eta = \eta(\mu, \lambda)$ be as in Lemma 4.3.16. By Lemma 4.3.22, there exist an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$:

$$\Phi_{n,\eta,K}(\mu,\lambda) = \mathbb{P}_\nu(L_n^\eta > K) \geq 1 - \Psi(\mu,\lambda) - \varepsilon/2.$$

Now let $n := \lfloor R/2\ln 2 \rfloor - 1$, and let F denote the event that $L_n^\eta > K$ (with respect to μ, λ). Given that F holds, we can pick $M = \Omega(K)$ intervals $I_1, \dots, I_M \subseteq \mathcal{U}_n$ of length $\eta 2^{-n}$ such that the angle between a point in I_i and a point in I_j is at least $1000 \cdot 2^{-n}$ (for all $1 \leq i \neq j \leq M$). Now let F_i denote the event that there is *exacltly one* point $X_\ell = (\rho_\ell, \theta_\ell) \in V_{\mathrm{Poi}}$ such that **1)** $R - \varepsilon < \rho_\ell \leq R$ and $\theta_\ell \in I_i$ and **2)** there is no point of $X_m = (\rho_m, \theta_m) \in \mathcal{P}_\mu$ with $\rho_m > r_n$ and θ_m within angle $10 \cdot 2^{-n}$ of one of the endpoints of I_i. Observe that

$$\mathbb{P}_\nu(F_i|F) = \mathbb{P}(\mathrm{Po}(\mu_1) = 1)\mathbb{P}(\mathrm{Po}(\mu_2) = 0) = \Theta(1),$$

where $\mu_1 := \eta \cdot 2^{-n} \cdot (\nu/4\pi) \cdot (\cosh(R/2) - \cosh((R-\varepsilon)/2))$ and $\mu_2 := 20 \cdot 2^{-n} \cdot (\nu/4\pi) \cdot (\cosh(R/2) - \cosh(r_n/2)) - \mu_1$. (That both μ_1, μ_2 are $\Theta(1)$ follows from the fact that $\cosh(R/2), \cosh((R-\varepsilon)/2), \cosh(r_n/2) = \Theta(2^n)$.) Note also that the event F_i-s are independent (given F). Hence we have

$$\mathbb{P}\left(\bigcup F_i|F\right) \geq 1-(1-\Theta(1))^M > 1-\varepsilon/2,$$

provided we chose K sufficiently large.

We now claim that, if F and some F_i hold, then there is a point $X_j \in W := \mathcal{P}_\mu \cap B_{\mathbb{H}}(O;R)$ that is at distance $> R$ from all other points in W (namely the sole vertex $X_j = (\rho_j, \theta_j)$ with angle in $\theta_j \in I_i$ and radius $\rho_j > R-\delta$). To see this, let $X_k = (\rho_k, \theta_k) \in W$ be an arbitrary other point. If $\rho_k > r_n$ we have $|\theta_j - \theta_k|_{2\pi} > 10 \cdot 2^{-n}$. On the other hand, we have $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) \leq \mathrm{dist}_{\mathbb{H}}(X_j', X_k')$ where $X_j' = (r_n, \theta_j), X_k' = (r_n, \theta_k)$ by Lemma 2.1.2. Hence, by the hyperbolic law of cosines $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) \leq R$ only if the difference in angle $|\theta_j - \theta_k|_{2\pi}$ is at most

$$\begin{aligned}\arccos\left(\frac{\cosh^2(r_n)-\cosh(R)}{\sinh^2(r_n)}\right) &= \arccos\left(1-O(e^{-r_n})\right) = (1+o(1))2e^{-r_n/2} \\ &= (1+o(1)) \cdot 2^{-(n-1)}.\end{aligned}$$

It follows $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) > R$.

Now suppose that $\rho_k < r_n$. Since $\theta_j \in \mathcal{U}_n$ it follows that

$$|\theta_j - \theta_k|_{2\pi} > (1+\delta)\arccos\left(\frac{\cosh(r_k)-1}{\sinh(r_k)}\right).$$

Now observe that, for $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) < R$ to hold, the angle between them can be at most $\arccos\left(\frac{\cosh(r_j)\cosh(r_k)-\cosh(R)}{\sinh(r_j)\sinh(r_k)}\right)$, by the hyperbolic law of cosines. Since $r_j \in (R-\varepsilon, R)$ we have that $\cosh(r_j) = (1+O(\varepsilon))\cosh(R)$ and $\sinh(r_j) = (1+O(\varepsilon))\cosh(R)$. This also gives that

$$\frac{\cosh(r_j)\cosh(r_k)-\cosh(R)}{\sinh(r_j)\sinh(r_k)} = (1+O(\varepsilon)\cdot\frac{\cosh(r_k)-1}{\sinh(r_k)}.$$

Using Taylor's expansion $\arccos(x+y) = arccos(x) + O(y/(1-x^2)^{1/2})$, we now find

$$\begin{aligned}
\arccos\left(\tfrac{\cosh(r_j)\cosh(r_k)-\cosh(R)}{\sinh(r_j)\sinh(r_k)}\right) &= \arccos\left(\tfrac{\cosh(r_k)-1}{\sinh(r_k)}\right) + O(\varepsilon e^{-r_k/2}) \\
&= (1+O(\varepsilon))\cdot\arccos\left(\tfrac{\cosh(r_k)-1}{\sinh(r_k)}\right).
\end{aligned}$$

(Using that $(\cosh(r_k)-1)/\sinh(r_k) = 1 - O(e^{-r})$. It follows that $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) > R$, as claimed. Hence if $(\bigcup F_j) \cap F$ has been realized, then at least one point of W will have distance larger than R to all other points of W.

We wish now to deduce that in such a case, $G(N; 1/2, \nu)$ will have an isolated vertex, but as it happens V_N is a strict subset of W. To get around this problem, we use the coupling from Lemma 4.4.1, and symmetry. Suppose that $(\bigcup F_j) \cap F$ holds, and choose a point X_j of distance $> R$ to all other points (uniformly at random from all such points, say). By symmetry considerations, under the coupling from Lemma 4.4.1 the probability that X_j is also a point of $\mathcal{P}_{\nu-\delta}$ is $\frac{\nu-\delta}{\nu+\delta} = 1 - O(\delta)$. Putting everything together, we find that

$$\begin{aligned}
& \mathbb{P}(G(N; 1/2, \nu) \text{ has an isolated vertex}) \\
\geq\ & \mathbb{P}\left(\bigcup F_i | F\right)\mathbb{P}_\nu(F) - O(\delta) - \mathbb{P}(\text{coupling fails}) \\
\geq\ & (1-\varepsilon/2)\cdot(1-\Psi(\mu,\lambda)-\varepsilon/2) - O(\delta) - o(1) \\
\geq\ & (1-\varepsilon/2)\cdot(1-\Psi(\nu,1)-\varepsilon) - O(\delta) - o(1).
\end{aligned}$$

Sending ε, δ to zero gives the lemma. □

To conclude, let us point out that Theorem 4.3.1 implies that $f(\nu) := \Psi(\nu, 1)$ has the properties described in Theorem 1.6.3 part 3.

Chapter 5

Distances

In this chapter we prove Theorems 1.6.4 and 1.6.5. Sections 5.1 and 5.2 give upper and lower bounds for the typical distance of any pair of connected vertices when a giant component is present, whereas we derive a bound for almost all pairs when no giant is present in Section 5.3.

5.1 Proof of Theorem 1.6.4: upper bound

We assume $1/2 < \alpha < 1$ for this section.

Definition 5.1.1. *For $G \in \mathcal{P}(N; \alpha, \nu)$ or $G \in \mathcal{G}(N; \alpha, \nu)$, let $\mathrm{Core}(G) = \{v \in V(G) : t_v \geq R/2\}$ be the core of G.*

Note that for every pair of vertices $u, v \in \mathrm{Core}(G)$, by the triangle inequality the distance between u and v is at most R, so $uv \in E(G)$. In other words, the subgraph that is induced by the vertices in $\mathrm{Core}(G)$ is complete.

Lemma 5.1.2. *Let $\omega(N)$ be such that $\omega(N) \to \infty$ as $N \to \infty$ but $\omega(N) = o(R)$. Let x be a vertex such that $t_x < \log\log R$ and $U \subset \mathcal{D}_R$ an open subset of $\mathcal{D}_R$ which does not contain any points of type at least $\log\log R$ and has $\mathrm{Area}_\alpha(U) = o(\mathrm{Area}_\alpha(\mathcal{D}_R))$. Let $G \in \mathcal{P}_{x,U}(N; \alpha, \nu)$. A.a.s. there is a vertex $u \in \mathrm{Core}(G)$ such that $uv \in E(G)$ for every vertex v with $t_v \geq \frac{2\alpha-1}{2\alpha}R + \omega(N)$.*

Proof. By the triangle inequality, any such vertex v is adjacent to any vertex of radius at most $R(2\alpha - 1)/(2\alpha) + \omega(N)$, so it is sufficient to show that a.a.s. the disc D_r of radius $r := \frac{2\alpha-1}{2\alpha}R + \omega(N)$ is non-empty. Note that $r < R/2$, for any N large enough, as $\omega(N) = o(R)$ and $\alpha < 1$, so any vertex in D_r belongs to the core. Let N_r be the number of vertices in D_r.

Note first that $\frac{2\alpha-1}{2\alpha} - 1 = -\frac{1}{2\alpha}$. Thus $r - R = -\frac{R}{2\alpha} + \omega(N)$, whereby $\alpha(r-R) = -R/2 + \alpha\omega(N)$. As $D_r \cap U = \emptyset$, these identities imply that

$$\begin{aligned}\mathbb{E}\left[\, N_r \,\right] &= (N-2)\frac{\cosh(\alpha r) - 1}{\mathrm{Area}_\alpha(\mathcal{D}_R) - \mathrm{Area}_\alpha(U)} = (N-2)\frac{\cosh(\alpha r) - 1}{\cosh(\alpha R)(1-o(1))} \\ &\sim N e^{\alpha(r-R)} = N e^{-R/2+\omega(N)} \overset{N=\nu e^{R/2}}{=} \nu e^{\alpha\omega(N)}.\end{aligned}$$

Using this and Fact 2.2.3 we get

$$\mathbb{P}(N_r \neq 0) = 1 - e^{-(1+o(1))\nu e^{\alpha\omega(N)}} = 1 - o(1).$$

$\square$

In fact, the only component we consider is the one containing the vertices in the core. We show that most pairs of vertices that are connected have a short path into the core. These naturally give short paths connecting all the vertices in the component. We are interested in the following paths in which the type of the vertices increases exponentially along the path.

Definition 5.1.3. *For $\delta > 0$, we call a path $P = v_1, v_2, \ldots, v_m$ in G a δ-*exploding path *if $v_m \in \mathrm{Core}(G)$ and $t_{v_{i+1}} \geq (1+\delta)t_{v_i}$ for $1 \leq i \leq m-2$.*

Not every vertex in the giant component has an exploding path into the core. However, the vertices that do not have such a path are more likely to have a very low type. In particular, we prove that any vertex of type at least $\log\log R$ has an exploding path into the core with probability $1 - o(1)$. We actually show this lemma for the Poisson model. The result does transfer to $\mathcal{G}(N;\alpha,\nu)$, due to its monotonicity, but we are going to use it later in this form.

Lemma 5.1.4. *Let $\delta = 2\frac{1-\alpha}{2\alpha-1}$ and $\zeta < \delta$ be a positive real number. Assume that v and x are vertices such that $t_v \geq \log\log R \geq t_x$ and $U \subset \mathcal{D}_R$ an open subset which does not contain any points of type at least t_v so that U is contained in a sector of $\mathcal{D}_R$ that spans a $o(1)$ angle. With probability (in the space $\mathcal{P}_{\{v,x\},U}(N;\alpha,\nu)$) $1 - e^{-\Theta(\log^{(\alpha-\frac{1}{2})\zeta} R)}$, there is a $(\delta-\zeta)$-exploding path starting at v.*

Proof. Take any $\varepsilon < \frac{1}{4}$ and assume that $N > N_0$, where N_0 is as in Lemma 2.1.7.

By Lemma 5.1.2, if v satisfies $t_v \geq \frac{2\alpha-1}{2\alpha}R + \omega(N)$, then a.a.s. there is a vertex $u \in G$ with $t_u \geq R/2$ and $vu \in E(G)$. In other words, if $t_v \geq \frac{2\alpha-1}{2\alpha}R + \omega(N)$, then we are done.

Assume now that $t_v < \frac{2\alpha-1}{2\alpha}R + \omega(N)$. As $1+\delta = \frac{1}{2\alpha-1}$, it follows that $(1+\delta)t_v < \frac{1}{2\alpha}R + \frac{\omega(N)}{2\alpha-1}$. Note that by Corollary 2.1.6, it suffices to consider only points of type no larger than $\frac{1}{2\alpha}R + \frac{\omega(N)}{2\alpha-1}$.

Let $v_1 = v$. We will construct inductively a series of (random) sets $T_i \subset \mathcal{D}_R$, for $i \geq 2$, in each of which we find a vertex v_i, which will be the ith vertex in the exploding path.

For two points p, p', let $\vartheta_{p,p'} = \theta_{p,p'}$ if p' is in the anti-clockwise direction from p, but $\vartheta_{p,p'} = -\theta_{p,p'}$, otherwise.

Assume that we have exposed v_i. For any point $p \in \mathcal{D}_R$ we let

$$\hat{T}_\varepsilon^-(p) := \left\{ p' \in \mathcal{D}_R \ : \ |t_{p'} - (1+\delta)t_p| < \zeta t_p, \ \frac{\varepsilon\nu}{N} e^{\frac{t_p + t_{p'}}{2}} \leq \vartheta_{p',p} \leq \frac{2(1-\varepsilon)\nu}{N} e^{\frac{t_{p'}+t_p}{2}} \right\}.$$

We take $T_i := \hat{T}_\varepsilon^-(v_i)$. Let A be the set of vertices that are located in $\hat{T}_\varepsilon^-(v_i)$. Note that, as the angle covered by U is $o(1)$, we have that $\mathrm{Area}_\alpha(U) = o(\mathrm{Area}_\alpha(\mathcal{D}_R))$. Hence, the area of a set in $\mathcal{D}_R \setminus U$ is within a $1-o(1)$ factor from the area in $\mathcal{D}_R$ (both on the hyperbolic plane of curvature $-\alpha^2$).

So, for any $\varepsilon \in (0, 1/4)$ and for N large enough we have

$$\begin{aligned}\mathbb{E}\left[\,|A|\,\right] &\geq 2(1-\frac{3}{2}\varepsilon)\frac{N-2}{2\pi}\int_{(1+\delta-\zeta)t_{v_i}}^{(1+\delta+\zeta)t_{v_i}} e^{\frac{1}{2}(t_{v_i}+t-R)}(1-o(1))e^{-\alpha t}dt \\ &\geq 2(1-\frac{3}{2}\varepsilon)(1-o(1))\frac{N}{2\pi}\frac{\nu}{N}e^{\frac{t_{v_i}}{2}}\int_{(1+\delta-\zeta)t_{v_i}}^{(1+\delta+\zeta)t_{v_i}} e^{(\frac{1}{2}-\alpha)t}dt \\ &\overset{\varepsilon<1/4}{\geq} \frac{\nu}{2\pi}e^{\frac{1}{2}t_{v_i}}\frac{1}{2\alpha-1}\left(e^{(\frac{1}{2}-\alpha)(1+\delta-\zeta)t_{v_i}} - e^{(\frac{1}{2}-\alpha)(1+\delta+\zeta)t_{v_i}}\right).\end{aligned}$$

But $(1+\delta+\zeta)t_{v_i} - (1+\delta)t_{v_i} + \zeta t_{v_i} > 2\zeta t_{v_i} \to \infty$, whereby the above becomes:

$$\mathbb{E}\left[\,|A|\,\right] \geq \frac{\nu}{2\pi}\frac{1}{2\alpha-1}e^{\frac{1}{2}t_{v_i}-(\alpha-\frac{1}{2})(1+\delta-\zeta)t_{v_i}}(1-o(1)).$$

Furthermore, $(\alpha-\frac{1}{2})(1+\delta) = \frac{2\alpha-1}{2}\frac{1}{2\alpha-1} = \frac{1}{2}$ and finally, we have

$$\mathbb{E}\left[\,|A|\,\right] \geq \frac{\nu}{2\pi}\frac{1}{2\alpha-1}e^{(\alpha-\frac{1}{2})\zeta t_{v_i}}(1-o(1)) \overset{2\alpha-1<1}{\geq} \frac{\nu}{2\pi}e^{(\alpha-\frac{1}{2})\zeta t_{v_i}},$$

for N large enough. Hence, by Fact 2.2.3 we have

$$\begin{aligned}\mathbb{P}(|A|>0) &= 1-\mathbb{P}(|A|=0) \\ &\geq 1-\exp\left(-\frac{\nu}{2\pi}e^{(\alpha-\frac{1}{2})\zeta t_{v_i}}\right).\end{aligned}$$

As $t_{v_i} \geq \log\log R$, we have $\mathbb{P}(|A|=0) \leq \exp\left(-\frac{\nu}{\pi}(\log R)^{(\alpha-\frac{1}{2})\zeta}\right)$. If $|A|>0$, then there are vertices that are located inside T_i and we let v_{i+1} be one of them – the choice is arbitrary. The following claim guarantees that $T_{i+1} = \hat{T}_\varepsilon^-(v_{i+1})$ is disjoint from T_i and when we repeat the argument there is no danger to expose again area which we have already exposed.

Claim 5.1.5. *For all N large enough and for all $i \geq 1$ the following holds. For all $p \in \hat{T}_\varepsilon^-(v_i)$ we have $T_\varepsilon^+(v_i) \cap \hat{T}_\varepsilon^-(p) = \emptyset$.*

Proof of Claim 5.1.5. Consider a point $p \in \hat{T}_\varepsilon^-(v_i)$ and let $p' \in \hat{T}_\varepsilon^-(p)$. We will show that

$$\vartheta_{p',v_i} \gg 2(1+\varepsilon)\frac{\nu}{N}e^{\frac{t_v+t_{p'}}{2}}.$$

We write $\vartheta_{p',v_i} = \vartheta_{p',p} + \vartheta_{p,v_i}$. Since $p' \in \hat{T}_\varepsilon^-(p)$ and $p \in \hat{T}_\varepsilon^-(v_i)$ we have

$$\vartheta_{p',p} \geq \varepsilon\frac{\nu}{N}e^{\frac{t_{p'}+t_p}{2}} \text{ and } \vartheta_{p,v_i} \geq \varepsilon\frac{\nu}{N}e^{\frac{t_p+t_{v_i}}{2}}.$$

Hence

$$\begin{aligned}
\vartheta_{p',p} + \vartheta_{p,v_i} &\geq \varepsilon\frac{\nu}{N}\left(e^{\frac{t_{p'}+t_p}{2}} + e^{\frac{t_p+t_{v_i}}{2}}\right) \\
&= \varepsilon\frac{\nu}{N}e^{\frac{t_{p'}+t_{v_i}}{2}}\left(e^{\frac{t_p-t_{v_i}}{2}} + e^{\frac{t_p-t_{p'}}{2}}\right) > \varepsilon\frac{\nu}{N}e^{\frac{t_{p'}+t_{v_i}}{2}}e^{\frac{t_p-t_{v_i}}{2}} \\
&\geq \varepsilon\frac{\nu}{N}e^{\frac{t_{p'}+t_{v_i}}{2}}e^{(\delta-\zeta)t_{v_i}} \overset{(\delta-\zeta)t_{v_i}\to\infty}{\gg} 2(1+\varepsilon)\frac{\nu}{N}e^{\frac{t_{p'}+t_{v_i}}{2}}.
\end{aligned}$$

In fact, $(\delta-\zeta)t_{v_i} \geq (\delta-\zeta)\log\log R$, and therefore the inequality holds uniformly for all N that are large enough. □

We now use Claim 5.1.5 to finish the proof of Lemma 5.1.4. If we start at type at least $\log\log R$, it takes $O(\log R)$ steps to reach type $\frac{2\alpha-1}{2\alpha}R + \omega(N)$; at that point we can complete the exploding path using the vertex whose existence is guaranteed by Lemma 5.1.2. Thus for any given vertex v with $t_v > \log\log R$ we have

$$\begin{aligned}
\mathbb{P}(\exists \text{ sequence of vertices } v_2, v_3, \ldots) &= \left(1 - \exp\left(-\frac{\nu}{\pi}(\log R)^{(\alpha-\frac{1}{2})\zeta}\right)\right)^{O(\log R)} \\
&= 1 - O(\log R)\exp\left(-\frac{\nu}{\pi}(\log R)^{(\alpha-\frac{1}{2})\zeta}\right) \\
&= 1 - \exp\left(-\Theta\left(\log^{(\alpha-\frac{1}{2})\zeta} R\right)\right),
\end{aligned}$$

as $xe^{-ax^b} = o(1)$ for $0 < a, b$ and $x \to \infty$.

□

Remark 5.1.6. *In fact, if the type of v is $O(1)$, which holds for all but $o(N)$ of the vertices, then the probability that there is a $(\delta-\zeta)$-exploding path starting at v is bounded away from 0. With slightly more work, one can show that two vertices u and v*

have both an exploding path with probability that is asymptotically bounded away from 0*. Thus,* $d_G(u,v) < \infty$ *with probability that is asymptotically bounded away from* 0*. Alternatively, this follows from Theorem 1.6.1.*

We are now ready to proceed with the upper bound in Theorem 1.6.4

Proof of Theorem 1.6.4: upper bound. Let u, v be two vertices. We will show that the event $d_G(u,v) < \infty$ but $d_G(u,v) \geq (2\tau + \zeta^{1/2}) \log R$ occurs with probability $o(1)$. Note that this is in the $\mathcal{G}(N;\alpha,\nu)$ space. Also, for convenience, we have taken the ζ that appears in the statement of Theorem 1.6.4 as $\zeta^{1/2}$. We denote this event by $\mathcal{E}_N(\tau,\zeta)$. Also, let $\mathcal{A}_N$ denote the event that the relative angle between u and v is greater than $\nu\frac{2\zeta_\varepsilon \log R}{N}$, where $\zeta_\varepsilon := \zeta(1-\varepsilon)$, for some $\varepsilon \in (0,1)$. Of course, the probability of $\overline{\mathcal{A}_N}$ is $o(1)$ and therefore it suffices to prove that $\mathbb{P}\left[\,\mathcal{E}_N(\tau,\zeta) \cap \mathcal{A}_N\,\right] = o(1)$.

If $\mathcal{E}_N(\tau,\zeta)$ is realised, then there must be a *minimal* path between vertices u and v. In this context, a minimal path is meant to be an induced path. Let $P_{\min}$ denote such a path. Assume, in addition, that $\mathcal{A}_N$ is simultaneously realised, that is, $\theta_{u,v} > \nu\frac{2\zeta_\varepsilon \log R}{N}$. With this assumption, let $P_{\min}(u)$ denote the sub-path of $P_{\min}$ starting at u and ending at the first vertex whose relative angle with u exceeds $\nu\frac{\zeta_\varepsilon \log R}{N}$. Similarly, let $P_{\min}(v)$ denote the sub-path of $P_{\min}$ starting at v and ending at the first vertex whose relative angle with v exceeds $\nu\frac{\zeta_\varepsilon \log R}{N}$. Clearly, as $\mathcal{A}_N$ is realized, the two paths may overlap, but they have at most one edge in common.

Assume without loss of generality that v is at angle $\theta_{u,v} \leq \pi$ in the anti-clockwise direction from u. Consider the sectors consisting of points of relative angle at most $\nu\frac{\zeta_\varepsilon \log R}{N}$ from a point x:

$$S_h^+(x) := \left\{ p \in \mathcal{D}_R \;:\; t_p > \log\log R,\ 0 < \vartheta_{x,p} < \nu\frac{\zeta_\varepsilon \log R}{N} \right\}$$

and

$$S_h^-(x) := \left\{ p \in \mathcal{D}_R \;:\; t_p > \log\log R,\ -\nu\frac{\zeta_\varepsilon \log R}{N} < \vartheta_{x,p} < 0 \right\}.$$

There are two cases:

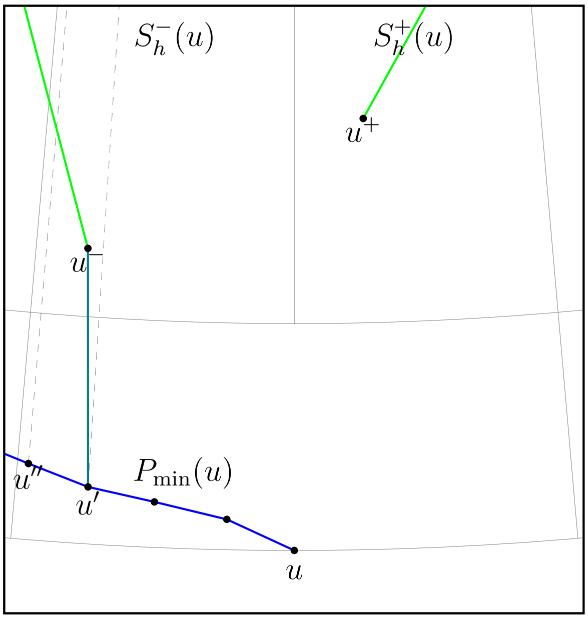

Figure 5.1: Creating a short path into the core under $\mathcal{S}$.

1. either each one of $S_h^+(u), S_h^-(u), S_h^+(v), S_h^-(v)$ contains a vertex that is the starting vertex of a $(\delta - \zeta)$-exploding path,

2. or at least one of them is either empty or all of its vertices are not the endpoints of a $(\delta - \zeta)$-exploding path.

Let $\mathcal{S}$ denote the former and let $\overline{\mathcal{S}}$ denote the latter. We will show that $\Pr(\overline{\mathcal{S}}) = o(1)$. First consider, without loss of generality, the set $S_h^+(u)$. The probability that this set is empty is $o(1)$. Indeed, let $N_{S_h^+(u)}$ be the number of vertices that appear in this sector. Then

$$\mathbb{E}\left[N_{S_h^+(u)}\right] = N\frac{\cosh(\alpha(R - \log\log R)) - 1}{\cosh(\alpha R) - 1}\,\frac{1}{2\pi}\,\nu\frac{\zeta_\varepsilon \log R}{N} \approx \log^{1-\alpha} R \to \infty.$$

The distribution of $N_{S_h^+(u)}$ is binomial and the application of a standard Chernoff bound implies that $\mathbb{P}\left[N_{S_h^+(u)} = 0\right] = o(1)$.

If $S_h^+(u)$ is not empty and all of its vertices are not the beginnings of a $(\delta - \zeta)$-exploding path, then the vertex with lowest type in $S_h^+(u)$ does not have a $(\delta - \zeta)$-exploding path starting at it as well. We call this vertex the *first* vertex in $S_h^+(u)$.

Claim 5.1.7. *The probability that the first vertex in $S_h^+(u)$ does not have a $(\delta-\zeta)$-exploding path starting at it is o(1).*

Proof of Claim 5.1.7. Conditional on having at least one vertex in $S_h^+(u)$, let u' be the first vertex (with probability 1 there will be exactly one such vertex) which we expose and assume that the area in $S_h^+(u)$ that consists of points with type greater than $t_{u'}$ has not been exposed. Let us switch temporarily to $\mathcal{P}_{X,U}(N;\alpha,\nu)$, where $X = u, u'$ and U the subset of $S_h^+(u)$ below u'. Then by Lemma 5.1.4, there is a $(\delta-\zeta)$-exploding path starting at u' with probability $1-o(1)$ uniformly over $t_{u'} \geq \log\log R$. This lemma can be applied as the area above u' has not been exposed in the corresponding Poisson process and the proof of Lemma 5.1.4 deals only only with that area. The result transfers to $\mathcal{G}(N;\alpha,\nu)$ (conditional on U being empty and on the realisations of u and u'), through Lemma 2.2.2, due to the fact that this property is non-decreasing. □

Then, since the probability that $S_h^+(u)$ is empty is $o(1)$, the union bound implies that $\mathbb{P}\left[\overline{\mathcal{S}}\right] = o(1)$.

We will show that $\mathbb{P}\left[\mathcal{E}_N(\tau,\zeta)\cap\mathcal{A}_N\cap\mathcal{S}\right] = 0$. Observe that any vertex which belongs to $S_h^+(u)\cup S_h^-(u)$ (or to $S_h^+(v)\cup S_h^-(v)$, respectively) will be adjacent to a vertex in $P_{\min}(u)$ ($P_{\min}(v)$, resp.). Indeed, if $P_{\min}(u)$ contains a vertex in $S_h^+(u)\cup S_h^-(u)$, then this must be adjacent to any other vertex in $S_h^+(u)\cup S_h^-(u)$. This is the case as $S_h^+(u)\cup S_h^-(u)\subseteq T_\varepsilon^-(u')$ for any $u'\in S_h^+(u)\cup S_h^-(u)$, provided that $\zeta<1$. To see this, note that any two points in $S_h^+(u)\cup S_h^-(u)$ have relative angle at most $2\zeta_\varepsilon\frac{\nu}{N}$. However, for any point in $S_h^+(u)\cup S_h^-(u)$, its inner tube consists of all points of relative angle at most $2(1-\varepsilon)\frac{\nu e^{\log\log R}}{N}$ from it. Thus, if $\zeta_\varepsilon<1-\varepsilon$ (that is, $\zeta<1$), then the containment follows. In this case, some vertex of $P_{\min}(u)$ will be connected to the first vertex in $S_h^+(u)\cup S_h^-(u)$.

Suppose now that all vertices of $P_{\min}(u)$ do not belong to $S_h^+(u)\cup S_h^-(u)$. Let u^+, u^- be vertices in $S_h^+(u)$ and $S_h^-(u)$ respectively, which are the starting vertices of $(\delta-\zeta)$-exploding paths P_{u^+} and P_{u^-}. There are two *consecutive* vertices in $P_{\min}(u)$ say u', u'' such that either $\vartheta_{u'',u^+}>0>\vartheta_{u',u^+}$ or $\vartheta_{u'',u^-}>0>\vartheta_{u',u^-}$. Thus, either u^+

or u^- is "above" the edge $u'u''$ and therefore by Fact 2.1.1 either u^+ or u^- is adjacent to both $u'u''$. The length of any exploding path is at most $\log R/\log(1+\delta-\zeta)$. Thus, $|P_{u^+}|, |P_{u^-}| \leq \log R/\log(1+\delta-\zeta)$. The following bounds the length of $P_{\min}(u), P_{\min}(v)$:

Claim 5.1.8. *Both $P_{\min}(u)$ and $P_{\min}(v)$ have length at most $\zeta \log R$.*

Proof of Claim 5.1.8. Consider $P_{\min}(u)$ (the proof for $P_{\min}(v)$ is identical). Since $P_{\min}(u)$ is part of a minimal path, it follows that if we take the set of vertices of $P_{\min}$ that are at even distance from u, then there cannot be an edge between any two of them, for this would contradict the minimality of $P_{\min}$. Let $P^e_{\min}(u)$ be this set of vertices. For any vertex $u' \in P^e_{\min}(u)$ consider the sector $T(u') := \{p \in \mathcal{D}_R \ : \ \theta_{u',p} < (1-\varepsilon)\frac{\nu}{N}\}$. There cannot be distinct $u', u'' \in P^e_{\min}(u)$ such that $T(u') \cap T(u'') \neq \emptyset$. If this were the case, then their relative angle would be at most $2(1-\varepsilon)\frac{\nu}{N}$ and by Lemma 2.1.7 they would be adjacent. But there are at most $\nu\frac{\zeta_\varepsilon \log R}{N} / \left(2(1-\varepsilon)\frac{\nu}{N}\right) = \frac{\zeta}{2}\log R$ such sectors inside the sector of angle $\nu\frac{\zeta_\varepsilon \log R}{N}$ in the anti-clockwise direction from u. Thus $|P^e_{\min}(u)| \leq \frac{\zeta}{2}\log R$, whereby the length of $P_{\min}$ is at most $\zeta \log R$. $\square$

Thus

$$
\begin{aligned}
d_G(u,v) &\leq |P_{\min}(u)| + |P_{u^+}| + 1 + |P_{u^-}| + |P_{\min}(v)| \\
&\leq 2\left(\frac{1}{\log(1+\delta-\zeta)} + \zeta + o(1)\right)\log R
\end{aligned}
$$

Hence, there exists a ζ such that for all N large enough $\frac{1}{\log(1+\delta-\zeta)} + \zeta + o(1) < \tau + \zeta^{1/2}$. This implies that $\mathcal{E}_N(\tau,\zeta)$ is not realised. $\square$

Remark 5.1.9. *If we replace the angles that determine the domains S_h^+ and S_h^- by a quantity that is proportional to $R^{\frac{1}{1-\alpha}}/N$ and the lower bound on the type by $\frac{1}{2(1-\alpha)}\log R$, then the probabilities that appear above become $o(N^{-2})$. Thus, the analogous of the above bound on $d_G(u,v)$ holds for all pairs of vertices, and implies that the diameter is proportional to $R^{\frac{1}{1-\alpha}}$ a.a.s. This upper bound is worse than the one obtained in [FK15].*

5.2 Proof of Theorem 1.6.4: lower bound

For given vertices $u, v \in \mathrm{V}_N$, let $\mathcal{L}_{\zeta,N}(u,v)$ be the event that $d_G(u,v) < (2\tau - \zeta)\log R =: L$, for some $\zeta > 0$. Assume that $t_u, t_v < \log\log R$ - by Lemma 2.1.5 this event occurs with probability $1 - o(1)$. Let $\mathcal{T}_{u,v}$ denote this event. By Lemma 2.1.7, for any $T \leq R/2 - 2\log\log R$, if u and v are connected through a path of length at most ℓ_u where the intermediate vertices have type at most T, then

$$\theta_{u,v} \leq 4\nu \frac{e^T}{N} L \leq 4\nu \frac{e^{R/2}}{N} \frac{L}{\log^2 R} = 4\frac{L}{\log^2 R}.$$

Conditional on $\mathcal{T}_{u,v}$, the probability of this event is $O(L/\log^2 R) = o(1)$. Now, if there is a path of length at most L that joins u to v that contains an intermediate vertex of type at least $R/2 - 2\log\log R$, then there must be a path of length at most $L/2$ either from u or from v to this vertex. Denote by $d_G(u, core)$ the graph distance of the vertex u to a vertex of type at least $R/2 - 2\log\log R$. The following lemma proves that almost all vertices are, in some sense, far away from vertices this type, immediately proving the lower bound.

Lemma 5.2.1. *Assume that $t_u \leq \log\log R$. For $\zeta > 0$, we have*

$$\Pr(d_G(u, core) \leq (\tau - \zeta^{1/2})\log R) = o(1).$$

We appeal to Lemma 2.2.2 on the event $\{d_G(u, core) \leq (\tau - \zeta^{1/2})\log R\}$. Clearly, this is a non-decreasing event in the sense that is used in that lemma. So, it suffices to prove Lemma 5.2.1 in the $\mathcal{P}_{\{u\},\emptyset}(N;\alpha,\nu)$ space.

To prove this statement, we keep track of the highest type in the neighbourhood of the vertex u, using a breadth exploration process as introduced in Chapter 2.3. Let $N^{(0)}(u) = \{u\}$, $\theta_r^{(0)} = \theta_\ell^{(0)} = 0$. For $i \geq 0$, define $N^{(i)}(u)$ as the neighbours of vertices in $N^{(i-1)}(u)$ that are in clockwise direction of u and have relative angle greater than $\theta_\ell^{(i-1)}$ with u or that are in anticlockwise direction of u and have relative angle with

u greater than $\theta_r^{(i-1)}$. Define $\theta_r^{(i)}$ as the maximum relative angle between u and any vertex in $N^{(i)}(u)$ that is in anticlockwise direction of u, setting it to $\theta_r^{(i-1)}$ if there is no such vertex. Similarly, define $\theta_\ell^{(i)}$ as the maximum relative angle between u and any vertex in $N^{(i)}(u)$ that is in clockwise direction of u, setting it to $\theta_\ell^{(i-1)}$ if there is no such vertex. This is the simultaneous breadth exploration process that will be defined in more detail in the next section.

Note that any vertex in $N^{(i)}(u)$ has graph distance i to u, but not every vertex of distance i is in $N^{(i)}(u)$. However, we claim that the process cannot leave a vertex that has type larger than the maximum type of any vertex in $N_i(u) := \bigcup_{j=0}^{i} N^{(j)}(u)$ and is within the sectors exposed undiscovered. For the sake of contradiction, assume that v is a vertex whose type is larger than the types of all vertices discovered in $N_i(u)$, but its angle with u satisfies $\theta_r^{(k-1)} < \vartheta_{u,v} \leq \theta_r^{(k)}$, for some $1 \leq k \leq i$. Then there are two vertices $v_{k-1} \in N^{(k-1)}(u)$ and $v_k \in N^{(k)}$ such that v is between them; that is, $\vartheta_{v,v_{k-1}} < 0 \leq \vartheta_{v,v_k}$. But the following holds (the second part will be used in the next section).

Claim 5.2.2. *Consider three vertices z, y and w, on $\mathcal{D}_R$ (in the hyperbolic plane with curvature -1), such that $d_H(z,w) < R$ and w is at the anticlockwise direction of z whereas y is between z and w. If $t_y > t_w$, then $d_H(y,z) < R$. Also, if $t_y > t_z$, then $d_H(y,w) < R$.*

Proof of Claim 5.2.2. This is the case as the point y' of type equal to that of y with $\theta_{y'w} = 0$ is still at distance less than R from z. If we move this clockwise towards z, the distance will remain smaller than R, as w will be at the anticlockwise side of y'. An analogous argument shows the second statement. □

The first part of the above claim with v_{k-1}, v, v_k playing the role of z, y, w implies that v is adjacent to v_{k-1} and therefore should have been discovered and become a member of $N^{(k)}(u)$.

The above claim has also the following consequence. Denote by $t^{(i-1)}$ the maximum type of a vertex in $N_{i-1}(u)$. As every vertex in $N^{(i-1)}(u)$ is further in the anticlockwise or in the clockwise direction, in terms of relative angle from u, than all the vertices in $N_{i-2}(u)$, all vertices in $N^{(i)}(u)$ are either within (hyperbolic) distance R and in the clockwise direction of the point $p_\ell^{(i-1)}$ of type $t^{(i-1)}$ and of clockwise relative angle $\theta_\ell^{(i-1)}$ to u, or within (hyperbolic) distance R and in the anticlockwise direction of the point $p_r^{(i-1)}$ of type $t^{(i-1)}$ and of clockwise relative angle $\theta_r^{(i-1)}$ to u. Thus the highest type of a vertex in $N^{(i)}(u)$ is stochastically dominated from above by the highest type among all vertices that have hyperbolic distance at most R from a certain point of type $t^{(i-1)}$ (namely $p_r^{(i-1)}$ or $p_\ell^{(i-1)}$). Due to this we can bound the distribution function of $t^{(i)}$ from below using Fact 2.2.3. Let $\hat{t}^{(i)} := (1+\delta+\zeta)^i t_u$, for any integer $i \geq 0$.

Claim 5.2.3. *For $i \geq 1$, assuming that $\hat{t}^{(i-1)} < \frac{R/2-2\log\log R}{1+\delta+\zeta}$, we have*

$$\Pr(t^{(i)} < (1+\delta+\zeta)\hat{t}^{(i-1)} \mid t^{(i-1)} < \hat{t}^{(i-1)}) \geq \exp\left(-\frac{2\nu}{(\alpha-1/2)\pi}e^{-(\alpha-1/2)\zeta\hat{t}^{(i-1)}}\right).$$

Proof. By the assumption of the claim, if $t^{(i-1)} < \hat{t}^{(i-1)}$, then $t^{(i-1)} < (1/(1+\delta+\zeta))(R/2-2\log\log R) < (2\alpha-1)R/2$. Lemma 2.1.7 works for types t such that $t + t^{(i-1)} < R - c_0$ for a given constant c_0, so $t < R-(1/(1+\delta))R/2$ will do. Recall that $1/(1+\delta) = 2\alpha-1$, so $t < R(3/2-\alpha)$ is sufficient. But $3/2-\alpha > 1/(2\alpha)$, and so if we take $\hat{t} = R/(2\alpha)+\omega(N)$, for some sufficiently slowly growing function $\omega(N)$, we are able to use Lemma 2.1.7 for points of type at most $\hat{t}$. The first part of Corollary 2.1.6 implies that the expected number of vertices of type at least $\hat{t}$ in $\mathcal{G}_{\{u\},\emptyset}(N;\alpha,\nu)$ is $o(1)$.

As discussed above, the event where $t^{(i)} \leq (1+\delta+\zeta)\hat{t}^{(i-1)}$ has no smaller probability than the event that a vertex of type $\hat{t}^{(i-1)}$ has no neighbour of type at least $\hat{t}^{(i)}$. Thus by Fact 2.2.3 and Lemma 2.1.7, for $\varepsilon > 0$ small enough so that $(1+2\varepsilon)\alpha < 1$ we have

$$
\begin{aligned}
&\Pr\left(t^{(i)} < (1+\delta+\zeta)\hat{t}^{(i-1)} \mid t^{(i-1)} < \hat{t}^{(i-1)}\right)\\
&\geq \exp\left(-N\int_{(1+\delta+\zeta)\hat{t}^{(i-1)}}^{\hat{t}} \frac{4(1+\varepsilon)}{2\pi} e^{1/2(t+\hat{t}^{(i-1)}-R)}\alpha e^{-\alpha t}dt + o(1)\right)\\
&\geq \exp\left(-\frac{2(1+2\varepsilon)\alpha\nu}{\pi} e^{\frac{\hat{t}^{(i-1)}}{2}} \int_{(1+\delta+\zeta)\hat{t}^{(i-1)}}^{\infty} e^{(1/2-\alpha)t}dt\right)\\
&\geq \exp\left(-\frac{2(1+2\varepsilon)\alpha\nu}{\pi} e^{\frac{\hat{t}^{(i-1)}}{2}} \frac{1}{\alpha-1/2} e^{(1/2-\alpha)(1+\delta+\zeta)\hat{t}^{(i-1)}}\right)\\
&\geq \exp\left(-\frac{2(1+2\varepsilon)\alpha\nu}{\pi} e^{\frac{\hat{t}^{(i-1)}}{2}} \frac{1}{\alpha-1/2} e^{(-1/2+(1/2-\alpha)\zeta)\hat{t}^{(i-1)}}\right)\\
&\geq \exp\left(-\frac{2\nu}{(\alpha-1/2)\pi} e^{-(\alpha-1/2)\zeta\hat{t}^{(i-1)}}\right),
\end{aligned}
$$

as $(\alpha-1/2)(1+\delta) = 1/2$. □

We repeatedly apply this bound to bound the distance from the core. Assume that $t_u = \log\log R$. Denote by $\mathcal{U}$ the event that if we explore as above the neighbours u for every $i < (\tau - \zeta^{1/2})\log R$ we have $t^{(i)} < \hat{t}^{(i)}$.

Claim 5.2.4. *Assume that $t_u = \log\log R$. For $\zeta > 0$ small enough (depending on α), the event $\mathcal{U}$ has probability $1 - o(1)$ and after the steps are completed the maximum type reached is less than $R/2 - 2\log\log R$, if N is sufficiently large.*

Proof. Conditional on the event U, after executing the $(\tau - \zeta^{1/2})\log R$ steps we have reached type less than

$$
\begin{aligned}
&(1+\delta+\zeta)^{(\tau-\zeta^{1/2})\log R}\log\log R = e^{\log(1+\delta+\zeta)(\tau-\zeta)\log R}\log\log R\\
&\leq R^{(\log(1+\delta)+\zeta)(\tau-\zeta^{1/2})}\log\log R\\
&= R^{(\tau^{-1}+\zeta)(\tau-\zeta^{1/2})}\log\log R\\
&= R^{1-\tau^{-1}\zeta^{1/2}+\tau\zeta-\zeta^{3/2}}\log\log R = o(R/2 - 2\log\log R).
\end{aligned}
$$

Moreover, we are able to apply Claim 5.2.3 repeatedly for this number of steps and deduce that $\mathcal{U}$ has probability

$$
\begin{aligned}
\Pr(\mathcal{U}) &\geq \prod_{i=0}^{(\tau-\zeta^{1/2})\log R} \exp\left(-\frac{2\nu}{(\alpha-1/2)\pi} e^{-(\alpha-1/2)\zeta(1+\delta+\zeta)^i \log\log R}\right) \\
&\geq \prod_{i=0}^{(\tau-\zeta^{1/2})\log R} \left(1-\frac{2\nu}{(\alpha-1/2)\pi} e^{-(\alpha-1/2)\zeta(1+\delta+\zeta)^i \log\log R}\right) \\
&\geq 1-\sum_{i=0}^{(\tau-\zeta^{1/2})\log R} \frac{2\nu}{(\alpha-1/2)\pi} e^{-(\alpha-1/2)\zeta(1+\delta+\zeta)^i \log\log R} \\
&\geq 1-\frac{4\nu}{(\alpha-1/2)\pi} e^{-(\alpha-1/2)\zeta \log\log R} = 1-o(1).
\end{aligned}
$$

□

Proof of Lemma 5.2.1. Fact 2.1.1 implies that increasing the type of a vertex will keep all edges intact, so any path will stay a path if we increase the type of one of its vertices. Thus by a simple coupling argument we have that $\Pr(d(u, core) \leq d | t_u) \leq \Pr(d(u, core) \leq d | t'_u)$ for $t_u \leq t'_u$. We can thus assume that $t_u = \log\log R$. By Claim 5.2.4, a.a.s. executing $(\tau - \zeta^{1/2})\log R$ steps yields maximum type that is less than $R/2 - 2\log\log R$, so

$$\Pr(d(u, core) \leq (\tau - \zeta^{1/2})\log R) = o(1).$$

□

5.3 Proof of Theorem 1.6.5

Here, we consider the case where $\alpha > 1$. In this case, by Theorem 1.6.1, all components contain at most a sublinear number of vertices. More precisely, we show that a.a.s. all components contain at most $N^{1/\alpha}$ vertices (up to a poly-logarithmic factor). In fact, there are many components of polynomial size (as there are many vertices of polynomial degree which do not belong to the same component).

To prove Theorem 1.6.5, for any given vertex we explore a path that in some sense

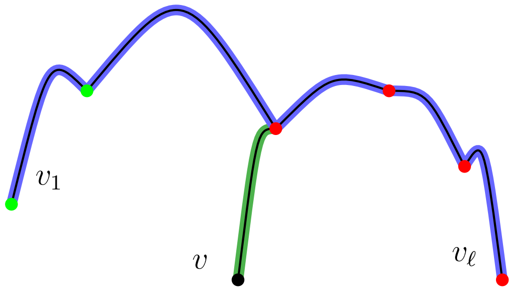

Figure 5.2: Example of an umbrella.

traverses its component. We show that almost all vertices are close to such a *spanning path*, which itself is short. This results in short distances for most pairs of vertices which belong to the same component.

Note that since $\alpha > 1$, a.a.s. there is no component whose convex hull contains the origin. In fact, components are included in a section of the disc spanning $o(1)$ of all angles. Due to this, it creates no ambiguity to talk of clockwise and anticlockwise directions in a component.

Definition 5.3.1. *We call a path $P = v_1, \ldots, v_\ell$ in a component C a* spanning path *of C if v_1 is the vertex of C that is farthest in clockwise and v_ℓ is the vertex of C that is farthest in anticlockwise direction.*

An umbrella *U with root vertex v is a spanning path P of the component of v together with a path connecting v to P. The* size *of the umbrella U is the maximum among the distances of v from the two endpoints of the associated spanning path.*

Note that any vertex in C that is above a spanning path P of C is directly connected to one of the vertices of P by Fact 2.1.1. Since there is no restriction on the length of the paths, if v is on some spanning path P, then P is an umbrella with root v.

The next result follows immediately as the vertices of a component that are to the farthest in clockwise and anticlockwise direction are always in a spanning path:

Corollary 5.3.2. *If P and P' are spanning paths of the same component, then $P \cap P' \neq \emptyset$.*

This fact allows us to do the following: Given any pair of vertices u and v in the same component, construct a u-v-path by traversing the umbrella U_u of u until the first vertex z that is on the umbrella U_v of v is reached. Then uU_uzU_vv is a path connecting u and v. Thus the following lemma is key to the proof of Theorem 1.6.5.

Lemma 5.3.3. *Let $\varepsilon > 0$. For a vertex v of $\mathcal{G}(N;\alpha,\nu)$, a.a.s. there is an umbrella for v of size at most $\log^{1+\varepsilon}\log N$.*

For the proof of this lemma we define the *simultaneous breadth exploration process* starting at a vertex v similar to the one introduced in Chapter 2. Here, we keep track of two sets of vertices V_ℓ and V_r, which both start out as $\{v\}$. Roughly speaking, we update the two sets adding the neighbours of the current sets that are located in the clockwise and anticlockwise direction from the "current" vertices, respectively. If there are no neighbours that are farther in the clockwise direction of V_r and no neighbours that are farther in the anticlockwise direction of V_ℓ, then the process stops. We define the process starting at vertex v as the following steps:

(i) Let $V_\ell^{(0)} = V_r^{(0)} = \{v\}$ and let $i := 1$.

(ii) Let $V_\ell^{(i)}$ be the set of vertices not in $V_\ell^{(i-1)} \cup V_r^{(i-1)}$ that are neighbours of some vertex in $V_\ell^{(i-1)} \cup V_r^{(i-1)}$ and are in the clockwise direction of every vertex in $\bigcup_{j=0}^{i-1}\{V_\ell^{(j)} \cup V_r^{(j)}\}$. We define similarly the set $V_r^{(i)}$ as the set of vertices not in $V_\ell^{(i-1)} \cup V_r^{(i-1)}$ that are neighbours of some vertex in $V_\ell^{(i-1)} \cup V_r^{(i-1)}$ and are in the anticlockwise direction of every vertex in $\bigcup_{j=0}^{i-1}\{V_\ell^{(j)} \cup V_r^{(j)}\}$.

(iii) If $V_\ell^{(i)} = \emptyset = V_r^{(i)}$, then stop. Otherwise, let $i := i + 1$ and go to step (ii).

We call a repetition of steps (ii) and (iii) a *round*. To prove Lemma 5.3.3, we show that this process yields an umbrella and bound the number of steps needed until completion.

Lemma 5.3.4. *If the simultaneous breadth exploration process starting at a vertex v stops after k rounds, then there is an umbrella for v that has size at most k.*

Proof. Let $C(v)$ denote the connected component that v belongs to. Let V_i' be the set of vertices discovered up to round i, that is $V_i' = \bigcup_{j=0}^{i}\{V_\ell^{(j)} \cup V_r^{(j)}\}$. We denote by v_ℓ' the vertex in V_i' with the largest relative angle with v in the clockwise direction. We let $\theta_\ell^{(i)}$ be this angle and let $t_\ell^{(i)}$ be the type of this vertex. Similarly, let v_r' be the vertex of V_i' that is the farthest in the anticlockwise direction, and let $\theta_r^{(i)}$ and $t_r^{(i)}$ denote its angle and type. Note that there is an edge between some vertex v_ℓ in V_{i-1}' to the vertex v_ℓ' in $V_\ell^{(i)}$ and also an edge between some vertex $v_r \in V_{i-1}'$ and the vertex v_r'.

We now claim that if the process stops at round k, then the vertices $\hat{v}_r$ and $\hat{v}_\ell$ that are the farthest to the anticlockwise and clockwise direction of $C(v)$ belong to V_{k-1}'. Note that $V_\ell^{(k)} = V_r^{(k)} = \emptyset$, so $V_{k-1}' = V_k'$. Assume this is not the case, so without loss of generality $\hat{v}_r \notin V_{k-1}'$. As v and $\hat{v}_r$ are in the same component, there is a path P from v to $\hat{v}_r$. Let w be the first vertex on P that is outside the range of angles from $\theta_\ell^{(k-1)}$ to $\theta_r^{(k-1)}$. Since $\hat{v}_r$ is the vertex that is farthest in the anticlockwise direction and $\hat{v}_r \notin V_k'$ this vertex must exist. Let u be the predecessor of w on P. We cannot have $u \in V_k'$ as otherwise w, being farther in the clockwise or anticlockwise direction than any other vertex in V_k', must also be in V_k' by the choice made in step (ii). There exists an $i < k$ and two *adjacent* vertices x and y such that x has been discovered at round $i-1$ and y has been discovered at round i and u is between x and y. Now, if $t_u \geq t_y$, then by Claim 5.2.2 (x, u, y playing the role of w, y, z) it follows that u is adjacent to x as well. If $t_u < t_y$, then again Claim 5.2.2 implies that y is adjacent to w. Hence, in either case w would have been discovered by round $i+1$, whereby $w \in V_r^{(i+1)} \cup V_\ell^{i+1} \subseteq V_k'$; a contradiction.

So both $\hat{v}_\ell$ and $\hat{v}_r$ are in V_k'. Note that every vertex in $V_\ell^{(i)} \cup V_r^{(i)}$ has a neighbour in $V_\ell^{(i-1)} \cup V_r^{(i-1)}$, so we can find a paths P_ℓ and P_r of length at most k from $\hat{v}_\ell$ to v and from $\hat{v}_r$ to v, respectively. Together, possibly deleting redundant subpaths in $v_\ell P_\ell v P_r v_r$, we have an umbrella for v of size at most k. □

We are now ready to prove Lemma 5.3.3

Proof of Lemma 5.3.3. We aim to bound the number of rounds it takes for the simultaneous breadth exploration process started at some vertex v to stop. By Corollary 2.1.6, it would be sufficient to consider a variation of the simultaneous breadth exploration process where we expose only those vertices that have type at most $R/(2\alpha) + \omega(N)$, for some slowly growing function $\omega(N) \to \infty$. We will use the same notation for the parameters of the process as in the unmodified process.

Let T denote the stopping time of this process. Without loss of generality, assume that $V_\ell^{(i)}, V_r^{(i)} \neq \emptyset$ for $i = 1, \ldots, T-1$. Define V_i', $\theta_\ell^{(i)}$ and $\theta_r^{(i)}$ as in the previous proof (but for the modified process). Unlike the last proof, let $t_\ell^{(i)}$ and $t_r^{(i)}$ be the maximum types of vertices in $V_\ell^{(i)}$ and $V_r^{(i)}$, respectively, and they are set to 0, if the corresponding set contains no vertices. Let $t_i = \max\{t_\ell^{(i)}, t_r^{(i)}\}$. Let $p_\ell^{(i)}$ be the point of type t_i and angle $\theta_\ell^{(i)}$ in the clockwise direction from v. Similarly, let $p_r^{(i)}$ be the point of type t_i and angle $\theta_r^{(i)}$ in the anticlockwise direction from v.

Claim 5.3.5. *We have* $V_\ell^{(i+1)} \subset T_\varepsilon^+(p_\ell^{(i)})$ *and* $V_r^{(i+1)} \subset T_\varepsilon^+(p_r^{(i)})$.

Proof of Claim 5.3.5. Let p be a point that is within hyperbolic distance R from $u \in V_\ell^{(i)} \cup V_r^{(i)}$ and satisfies $\vartheta_{p,v} > \theta_\ell^{(i)}$. Let u' be the point of type $t_\ell^{(i)}$, which has $\theta_{u,u'} = 0$.

Note that $\vartheta_{p,p_\ell^{(i)}} \leq \vartheta_{p,u}$. Since $p \in T_\varepsilon^+(u)$, we have $\vartheta_{p,u} \leq 2(1+\varepsilon)\frac{\nu}{N} e^{\frac{t_p + t_u}{2}}$. As $t_u \leq t_{u'} = t_\ell^{(i)}$, it follows that $\vartheta_{p,u} \leq 2(1+\varepsilon)\frac{\nu}{N} e^{\frac{t_p + t_\ell^{(i)}}{2}}$. In other words, $p \in T_\varepsilon^+(p_\ell^{(i)})$. Thereby, $V_\ell^{(i+1)} \subset T_\varepsilon^+(p_\ell^{(i)})$.

The proof that $V_r^{(i+1)} \subset T_\varepsilon^+(p_r^{(i)})$ is analogous. $\square$

The above claim implies that the highest type of a vertex in $V_\ell^{(i+1)}$, which we denoted by $t_\ell^{(i)}$, is stochastically dominated by the highest type among the vertices in $\left\{p \in T_\varepsilon^+(p_\ell^{(i)}) \ : \ \vartheta_{p,p_\ell^{(i)}} > 0,\ t_p < R/(2\alpha) + \omega(N)\right\}$. Similarly, the highest type of a vertex in $V_r^{(i+1)}$, which we denoted by $t_r^{(i)}$ is stochastically dominated by the highest type among the vertices in $\left\{p \in T_\varepsilon^+(p_r^{(i)}) \ : \ \vartheta_{p,p_r^{(i)}} < 0,\ t_p < R/(2\alpha) + \omega(N)\right\}$. Let $T_\ell(p_\ell^{(i)})$ and $T_r(p_r^{(i)})$ denote these two sets.

Thus, t_{i+1} is stochastically bounded from above by the largest type in $T_\ell(p_\ell^{(i)}) \cup T_r(p_r^{(i)})$. In turn, this is stochastically bounded from above by the maximum type of a vertex in $T_\ell(p^{(i)}) \cup T_r(p^{(i)})$ for a point $p^{(i)}$ of type $t_i = \max\{t_\ell^{(i)}, t_r^{(i)}\}$. We shall proceed with the estimation of the cdf of the latter random variable.

Observe first that Claim 5.3.5 implies that for all $0 < i \leq T$ we have $V_i' \subset \bigcup_{j=0}^{i-1}\{T_\varepsilon^+(p_\ell^{(j)}) \cup T_\varepsilon^+(p_r^{(j)})\}$, assuming that $p_\ell^{(0)}, p_r^{(0)}$ are both set to the point of $\mathcal{D}_R$ where v is located. Let $\mathcal{N}_i$ be the set of vertices that belong to V_i'. For a vertex $u \in \mathrm{V}_N \setminus V_i'$, the distribution on $\mathcal{D}_R$ is uniform (within the plane of curvature $-\alpha^2$) on the subset of $\mathcal{D}_R$ that excludes the union of the balls of radius R around each vertex in V_i'. Recall that $\mathrm{Area}_\alpha(\cdot)$ denotes the area of a measurable subset of $\mathcal{D}_R$ on the hyperbolic plane of curvature $-\alpha^2$. By Lemma 2.1.7 and the above observation, the area of the latter is at most $\sum_{j=0}^{i-1} \mathrm{Area}_\alpha(T_\varepsilon^+(p_\ell^{(j)}) \cup T_\varepsilon^+(p_r^{(j)}))$. But for each j, the angle that is spanned by $T_\varepsilon^+(p_\ell^{(j)}) \cup T_\varepsilon^+(p_r^{(j)})$ is proportional to $e^{R/(2\alpha)-R+\omega(N)} = o(1)$. Thus, if $i < R$, then we have $\sum_{j=0}^{i-1} \mathrm{Area}_\alpha(T_\varepsilon^+(p_\ell^{(j)}) \cup T_\varepsilon^+(p_r^{(j)})) = o(\mathrm{Area}_\alpha(\mathcal{D}_R))$.

Using this, we conclude that the conditional probability that a vertex $u \in \mathrm{V}_N \setminus \mathcal{N}_i$ belongs to $T_\varepsilon^+(p^{(i)})$ and has type t_u that satisfies $t \leq t_u < R/(2\alpha) + \omega(N)$ is at most

$$\begin{aligned}
&\int_t^{\frac{R}{2\alpha}+\omega(N)} \frac{4(1+\varepsilon)}{2\pi} e^{\frac{t_i+t'-R}{2}} \frac{\alpha \sinh(\alpha(R-t'))}{\cosh(\alpha R)(1-o(1))} dt' \\
&\leq \frac{2\alpha(1+2\varepsilon)}{\pi} e^{\frac{t_i-R}{2}} \int_t^{\frac{R}{2\alpha}+\omega(N)} e^{t'/2} \frac{e^{\alpha(R-t')}}{2\cosh(\alpha R)(1-o(1))} dt' \\
&\leq \frac{2\alpha(1+3\varepsilon)}{\pi} e^{\frac{t_i-R}{2}} \int_t^{\frac{R}{2\alpha}+\omega(N)} e^{\left(\frac{1}{2}-\alpha\right)t'} dt' \\
&= \frac{2\alpha\nu(1+3\varepsilon)}{\pi} \frac{e^{t_i/2}}{N} \int_t^{\frac{R}{2\alpha}+\omega(N)} e^{\left(\frac{1}{2}-\alpha\right)t'} dt' < \frac{4\alpha\nu(1+3\varepsilon)}{\pi(2\alpha-1)} \frac{e^{t_i/2}}{N} e^{\left(\frac{1}{2}-\alpha\right)t},
\end{aligned}$$

for N sufficiently large. Therefrom, the conditional probability that *none* of the vertices in $\mathrm{V}_N \setminus \mathcal{N}_i$ satisfies this is at least

$$\begin{aligned}
&\left(1 - \frac{4\alpha\nu(1+3\varepsilon)}{\pi(2\alpha-1)} \frac{e^{t_i/2}}{N} e^{\left(\frac{1}{2}-\alpha\right)t}\right)^{|\mathrm{V}_N \setminus \mathcal{N}_i|} > \left(1 - \frac{4\alpha\nu(1+3\varepsilon)}{\pi(2\alpha-1)} \frac{e^{t_i}/2}{N} e^{\left(\frac{1}{2}-\alpha\right)t}\right)^N \\
&> \exp\left(-D_{\alpha,\nu,\varepsilon} e^{\frac{t_i}{2}-(\alpha-1/2)t}\right),
\end{aligned} \tag{5.1}$$

for some $D_{\alpha,\nu,\varepsilon} > 0$ and any N sufficiently large. Therefore, for $i < R$ the random variable $\max\{t_\ell^{(i+1)}, t_r^{(i+1)}\}$ conditional on the history of the process up to step i is stochastically dominated by a random variable that follows the Gumbel distribution (introduced by and named after Gumbel [Gum35]). The expectation of the latter is

$$\frac{t_i + 2\ln(2D_{\alpha,\nu,\varepsilon})}{2\alpha - 1} + \frac{2\gamma}{2\alpha - 1},$$

where γ is Euler's constant. Therefore, the following inequality holds:

$$\mathbb{E}\left[\, t_{i+1} | \mathcal{F}_i \,\right] \leq \frac{t_i + 2\ln(2D_{\alpha,\nu,\varepsilon})}{2\alpha - 1} + \frac{2\gamma}{2\alpha - 1},$$

where $\mathcal{F}_i$ denotes the sub-σ-algebra generated by the process up to step i. There exists a constant $U_{\alpha,\nu,\varepsilon} > 0$ such that when $t_i > U_{\alpha,\nu,\varepsilon}$, we have

$$\mathbb{E}\left[\, t_{i+1} | \mathcal{F}_i \,\right] \leq \frac{t_i + 2\ln(2D_{\alpha,\nu,\varepsilon})}{2\alpha - 1} + \frac{2\gamma}{2\alpha - 1} < \frac{\alpha}{2\alpha - 1} t_i =: \lambda_\alpha t_i < t_i. \tag{5.2}$$

On the other hand, (5.1) implies that if $t_i \leq U_{\alpha,\nu,\varepsilon}$, then

$$\mathbb{P}(t_{i+1} = 0) \geq p > 0, \tag{5.3}$$

for some positive constant p.

With these tools, we can bound the stopping time T of the process. Let $[T_1^{(s)}, T_2^{(s)} \wedge R]$ denote the sth interval of indices in which the process stays above $U_{\alpha,\nu,\varepsilon}$. By (5.2), for $T_1^{(s)} < i \leq T_2^{(s)} \wedge R$ the process (t_i) is a supermartingale with decay rate at most λ_α.

Claim 5.3.6. *For any $\varepsilon' > 0$*

$$\Pr((T_2^{(s)} \wedge R) - T_1^{(s)} \geq \log_{1/\lambda_\alpha}^{1+\varepsilon'} R) = o(1).$$

Proof of Claim 5.3.6. Let $S := \log_{1/\lambda_\alpha}^{1+\varepsilon'} R$ and let $T^{(s)} := T_2^{(s)} \wedge R$. Note that we have $\mathbb{E}\left[t_{i\wedge T^{(s)}} \mid \mathcal{F}_{T_1^{(s)}}\right] \le \lambda_\alpha^{i\wedge T^{(s)}-T_1^{(s)}} t_{T_1^{(s)}} \le \lambda_\alpha^{i\wedge T^{(s)}-T_1^{(s)}} R$. Let A be the event $\{T^{(s)} > S+T_1^{(s)}\}$. If $\omega \in A$, then $\lambda_\alpha^{(S+T_1^{(s)}(\omega))\wedge T^{(s)}(\omega)-T_1^{(s)}} t_{T_1^{(s)}}(\omega) < \lambda_\alpha^S R = o(1)$. By the definition of the conditional expectation, we deduce that $\mathbb{E}\left[t_{(S+T_1^{(s)})\wedge T^{(s)}}\mathbf{1}_A\right] = o(1)$ and since $\mathbb{E}\left[t_{(S+T_1^{(s)})\wedge T^{(s)}}\mathbf{1}_A\right] > U_{\alpha,\nu,\varepsilon}\Pr(A)$, we finally deduce that $\Pr(A) = o(1)$. $\square$

Now, the length of the (discrete) interval $(T_2^{(s)}, T_1^{(s+1)} \wedge T \wedge R)$ is stochastically bounded from above by a geometric random variable that has parameter at least p.

We call the union of these intervals an *epoch*, that is, we call an epoch the interval $[T_1^{(s)}, T_1^{(s+1)} \wedge T \wedge R)$, for some $s > 0$. By the above claim, for any $\varepsilon' > 0$, with probability $1 - o(1)$, we have $(T_2^{(s)} \wedge R) - T_1^{(s)} \le \log_{1/\lambda_\alpha}^{1+\varepsilon'} R$. Additionally, the stochastic upper bound on the interval $(T_2^{(s)}, T_1^{(s+1)})$ implies that this is at most $\log_{1/\lambda_\alpha}^{\varepsilon'} R$ with probability $1 - o(1)$. Hence, with probability $1 - o(1)$ an epoch lasts for at most $\log_{1/\lambda_\alpha}^{1+2\varepsilon'} R$ steps. Finally, since every epoch has probability at least p to be the final one, it follows that the process hits 0 within $\log_{1/\lambda_\alpha}^{1+3\varepsilon'} R$ steps with probability $1 - o(1)$. In other words, a.a.s. we have $T \le \log_{1/\lambda_\alpha}^{1+3\varepsilon'} R$.

$\square$

Using the previous lemmas we prove Theorem 1.6.5.

Proof of Theorem 1.6.5. Let $0 < \varepsilon' < \varepsilon$. Let V' be the set of vertices in $\mathcal{G}(N;\alpha,\nu)$ that have an umbrella of size at most $\log^{1+\varepsilon'}\log N$. By Lemma 5.3.3 we have $|V'| = (1 - o(1))N$ a.a.s. For any $u, v \in V'$, if they are in the same component, by Corollary 5.3.2 the umbrellas are not disjoint. Thus there is a u-v-path of length at most $|U_u| + |U_v| \le 2\log^{1+\varepsilon'}\log N < \log^{1+\varepsilon}\log N$ for N large enough. $\square$

Chapter A

Appendix

A.1 Proof of Lemma 1.4.1

Proof of Lemma 1.4.1. Note that $R = (2/\zeta)\log(N/\nu)$ and $R' := (2/\zeta')\log(N/\nu)$ are chosen such that $N = \nu e^{\zeta R/2} = \nu e^{\zeta' R'/2}$.

The desired coupling is constructed as follows. We pick $\theta_1, \dots, \theta_N$ i.i.d. uniform on $[0, 2\pi)$ and we pick $U_1, \dots, U_N$ i.i.d. uniform on $[0, 1]$.

We now let $\rho_1, \dots, \rho_N$ and $\rho_1', \dots, \rho_N'$ be defined by the equations:

$$F_{\alpha,R}(\rho_i) = F_{\alpha',R'}(\rho_i') = U_i \qquad \text{(for } i = 1, \dots, N,\text{)} \tag{A.1}$$

where $F_{\alpha;R}$ is the cdf that goes with the pdf (1.1). That is:

$$F_{\alpha;R}(r) = \begin{cases} 0 & \text{if } r < 0, \\ \frac{\cosh(\alpha r)-1}{\cosh(\alpha R)-1} & \text{if } 0 \le r \le R; \\ 1 & \text{otherwise.} \end{cases} \tag{A.2}$$

(Note that in this way the ρ_is have exactly the distribution with pdf (1.1) and the ρ_i's have the same pdf but with α', R' in place of α, R.) The points used in the construction of $G(N; \zeta, \alpha, \nu)$ will be $(\theta_1, \rho_1), \dots, (\theta_N, \rho_N)$ while the points used in the construction of $G(N; \zeta', \alpha', \nu)$ will be $(\theta_1, \rho_1'), \dots, (\theta_N, \rho_N')$.

It remains to be seen that this way we get two isomorphic graphs.

Claim A.1.1. *We have $\rho_i' = (\alpha/\alpha')\rho_i$ for all i.*

Proof. Observe that

$$\alpha' R' = \alpha' \cdot (\zeta/\zeta')R = \alpha' \cdot (\alpha/\alpha')R = \alpha R.$$

Thus, the equation (A.1) defining ρ_i and ρ_i' yields:

$$\cosh(\alpha\rho_i) = \cosh(\alpha'\rho_i').$$

Since $\cosh(x)$ is strictly increasing for $x \geq 0$, it follows that we must have $\alpha\rho_i = \alpha'\rho_i'$. □

Let us write d_{ij} for the distance between (θ_i, ρ_i) and (θ_j, ρ_j) in the curvature-ζ-surface, and let d_{ij}' be defined analogously.

Claim A.1.2. *For all i, j we have $d_{ij}' = (\alpha/\alpha')d_{ij}$.*

Proof. By the *hyperbolic cosine rule* we have that

$$\cosh(\zeta d_{ij}) = \cosh(\zeta\rho_i)\cosh(\zeta\rho_j) - \sinh(\zeta\rho_i)\sinh(\zeta\rho_j)\cos(|\theta_i - \theta_j|),$$

and

$$\cosh(\zeta' d_{ij}') = \cosh(\zeta'\rho_i')\cosh(\zeta'\rho_j') - \sinh(\zeta'\rho_i')\sinh(\zeta'\rho_j')\cos(|\theta_i - \theta_j|).$$

Now observe that

$$\zeta\rho_i = \alpha \cdot (\zeta/\alpha) \cdot \rho_i = \alpha \cdot (\zeta'/\alpha')\rho_i = \zeta'\rho_i',$$

using Claim A.1.1, and similarly $\zeta\rho_j = \zeta'\rho_j'$. It follows that

$$\cosh(\zeta' d_{ij}') = \cosh(\zeta d_{ij}).$$

Again using that $\cosh(x)$ is strictly increasing for $x \geq 0$ (and the distances d_{ij}, d'_{ij} are nonnegative), we see that $d'_{ij} = (\zeta/\zeta')d_{ij} = (\alpha/\alpha')d_{ij}$. □

Since $R' = (\zeta/\zeta')R = (\alpha/\alpha')R$, we see that

$$d_{ij} \leq R \text{ if and only if } d'_{ij} \leq R',$$

which proves the lemma. □

A.2 The proof of Lemma 1.4.2

Very similarly to the proof of Lemma 1.4.1, the coupling is constructed as follows. We pick $\theta_1, \dots, \theta_N$ i.i.d. uniform on $[0, 2\pi)$ and we pick $U_1, \dots, U_N$ i.i.d. uniform on $[0, 1]$. We now let $\rho_1, \dots, \rho_N$ and $\rho'_1, \dots, \rho'_N$ be defined by the equations:

$$F_{\alpha,R}(\rho_i) = F_{\alpha',R'}(\rho'_i) = U_i \qquad \text{(for } i = 1, \dots, N.) \tag{A.3}$$

(Here $F_{\alpha,R}$ is as defined in the proof of Lemma 1.4.1, and $R := 2\log(N/\nu), R' := 2\log(N/\nu')$.) Again, we note that in this way the ρ_is have exactly the distribution with cdf $F_{\alpha,R}$ and the ρ'_is have cdf $F_{\alpha',R'}$. The points used in the construction of $G(N; \alpha, \nu)$ will be $(\rho_1, \theta_1), \dots, (\rho_N, \theta_N)$ while the points used in the construction of $G(N; \alpha', \nu')$ will be $(\rho'_1, \theta_1), \dots, (\rho'_N, \theta_N)$.

We need the following geometric fact.

Lemma A.2.1. *Suppose that $p = (r, \theta), q = (s, \vartheta)$ are two points in the hyperbolic plane satisfying $\mathrm{dist}_{\mathbb{H}}(p, O), \mathrm{dist}_{\mathbb{H}}(q, O), \mathrm{dist}_{\mathbb{H}}(p, q) \leq R$ and let $p' = (r', \theta), q' = (s', \vartheta)$ with $r' \leq r, s' \leq s$. Then $\mathrm{dist}_{\mathbb{H}}(p', q') \leq R$.*

Before giving the proof of this lemma, let us remind the reader that disks are convex, also in the hyperbolic plane. This means that if D is a disk in the hyperbolic plane and $x, y \in D$ then the geodesic between x, y is contained in D. One way to see this is by noting that every disk can be isometrically mapped to a disk with origin O, and that in the projective disk model of the hyperbolic plane (a.k.a. the Beltrami-Klein model) a hyperbolic disk with origin O looks like a Euclidean disk, while geodesics are just line segments in the projective disk model. (See for instance Section 4.8 of [Sti92] for a description of the projective disk model.)

Proof of Lemma A.2.1: It is enough to consider the case when $r' < r$ and $s' = s$. (Another application of this case will then give the full result.) Observe that the geodesic between O and p is just the line segment between them. So in particular, p' lies on the geodesic between O and p. Since $O, p \in B(q; R)$ it follows that also $p' \in B(q; R)$, as required. ∎

We also need the following observation, which can be rephrased as stating that the radius under the (α, R)-quasi uniform distribution stochastically dominates the radius under the (α', R)-quasi uniform distribution if $\alpha > \alpha'$.

Lemma A.2.2. *If $\alpha \geq \alpha'$ and $\nu = \nu'$ then $F_{\alpha,R}(r) \leq F_{\alpha',R'}(r)$ for every $r \in \mathbb{R}$.*

Proof. Note that $\nu = \nu'$ implies that also $R = R'$. Let us thus fix $R > 0$ and $0 < r < R$, and define $\varphi(\alpha) := F_{\alpha,R}(r)$ for every $\alpha > 0$. Our aim will be to show that $\frac{\mathrm{d}\varphi}{\mathrm{d}\alpha}$ is non-positive for every $\alpha > 0$, which will clearly yield the result.

We obtain:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\alpha} = \frac{r\sinh(\alpha r)(\cosh(\alpha R) - 1) - R\sinh(\alpha R)(\cosh(\alpha r) - 1)}{(\cosh(\alpha R) - 1)^2}.$$

Observe that this is non-positive if and only if

$$\frac{\alpha R\sinh(\alpha R)}{\cosh(\alpha R) - 1} \geq \frac{\alpha r\sinh(\alpha r)}{\cosh(\alpha r) - 1}.$$

We claim this is the case for all $0 \leq r \leq R$. To see this, it suffices to show that $(x\sinh x)/(\cosh x - 1)$ is nondecreasing for $x \geq 0$. Let us thus compute

$$\begin{aligned}
\left[\frac{x\sinh x}{\cosh x - 1}\right]' &= \frac{(\sinh x + x\cosh x)(\cosh x - 1) - x\sinh^2 x}{(\cosh x - 1)^2} \\
&= \frac{\sinh x\cosh x + x\cosh^2 x - \sinh x - x\cosh x - x\sinh^2 x}{(\cosh x - 1)^2} \\
&= \frac{\sinh x\cosh x + x(\cosh^2 x - \sinh^2 x) - \sinh x - x\cosh x}{(\cosh x - 1)^2} \\
&= \frac{\sinh x\cosh x + x - \sinh x - x\cosh x}{(\cosh x - 1)^2} \\
&= \frac{(\sinh x - x)(\cosh x - 1)}{(\cosh x - 1)^2} \\
&\geq 0.
\end{aligned}$$

So our claim holds, and we see that indeed $\frac{\mathrm{d}\varphi}{\mathrm{d}\alpha} \leq 0$ for all $\alpha > 0$. This proves the lemma. $\square$

Lemma A.2.3. *If $\alpha = \alpha'$ and $\nu \leq \nu'$, then $F_{\alpha,R}(r) \leq F_{\alpha',R'}(r)$ for every $r \in \mathbb{R}$.*

Proof. Observe that $\nu \leq \nu'$ implies that $R \geq R'$. This also gives $\cosh(\alpha R) - 1 \geq \cosh(\alpha R') - 1$, and hence the lemma. $\square$

Combining the last two lemmas gives:

Corollary A.2.4. *If $\alpha \geq \alpha'$ and $\nu \leq \nu'$, then $F_{\alpha,R}(r) \leq F_{\alpha',R'}(r)$ for every $r \in \mathbb{R}$.*

Together with the definition of ρ_i, ρ_i' this immediately gives:

Corollary A.2.5. *If $\alpha \geq \alpha'$ and $\nu \leq \nu'$, then, in the coupling described above, we have that $\rho_i \geq \rho_i'$ for all $1 \leq i \leq N$.*

Corollary A.2.5 together with Lemma A.2.1 yield Lemma 1.4.2.

A.3 Source Code

In this section, we give the Matlab code for the simulation of KPKVB graphs. The following script samples a graph with the given parameters and draws it, as well as the distribution of radii and degrees.

```
clear all;
close all;
N=1000;
alpha=1.2;
nu=1;
R=2*log(N/nu);
cosh_R = cosh(R);
V = rand(N, 2);

RAM=550000000;

r=acosh(V(:,1)*(cosh(alpha*R)-1)+1)/alpha;
theta=V(:,2)*2*pi;
cosh_r = cosh(r);
cos_theta = cos(theta);
sin_theta = sin(theta);
r_S=sqrt((cosh_r-1)./(cosh_r+1));
x_S=r_S.*cos_theta;
y_S=r_S.*sin_theta;
norm=x_S.^2+y_S.^2;
x=r.*cos_theta;
y=r.*sin_theta;
coords = [x,y];

S_help1 = [2*x_S.^2, 2*x_S, 2*ones(N,1), 2*y_S.^2,
    2*y_S, 2*ones(N,1)];
S_help1 = bsxfun(@rdivide, S_help1, 1-norm);
S_help1 = [ones(N,1), S_help1];
```

```
S_help2 = [ones(N,1), -2*x_S, x_S.^2 , ones(N,1),
    -2*y_S, y_S.^2 ];
S_help2 = bsxfun(@rdivide, S_help2, 1-norm);
S_help2 = [ones(N,1), S_help2];

connMatrix = sparse(1:N,1:N,false);
inc=ceil(RAM/N);
i=0;
fprintf('<');
while i+inc<N
    temp=sparse(S_help1(i+1:i+inc,:)*S_help2(i+1:end,:)'
                <cosh_R);
    connMatrix(i+1:i+inc,i+1:end) = temp;
    connMatrix(i+inc+1:end,i+1:i+inc) = temp(:,inc+1:end)';
    i=i+inc;
    inc=ceil(RAM/(N-i));
end
connMatrix(i+1:end,i+1:end)=
    S_help1(i+1:end,:)*S_help2(i+1:end,:)'<cosh_R;

save(['results.mat'],'-v7.3');
h = figure(1);
hold on;
t=linspace(0,2*pi);
plot(R*cos(t),R*sin(t),'Color',[0.6 0.6 0.6])
gplot(connMatrix,coords,'.-b')
scatter(coords(:,1), coords(:,2), 9,[0,0,0],'filled')
title('graph');
```

```
axis equal;
axis off;
hold off;
saveas(h, ['results.pdf']);
saveas(h, ['results.fig']);

h1 = figure(2);
degrees = sum(connMatrix)-1;
hist(degrees,min(degrees):max(degrees));
title('degree histogram')
saveas(h1, ['resultsdegrees.pdf']);
```

The next four programs are used to approximate the function $f(\nu)$ as it appears in Theorem 1.6.3. The function `isconnected.m` determines whether a graph, given as an adjacency matrix, is connected, using a depth first search to uncover the connected component of the first vertex.

```
function S = isconnected(adj)
d=dfs(adj,1);
S=true;
if(min(d)==-1)
S=false;
end
end
```

The function `connected.m` creates a sample of $\mathcal{G}(N;\alpha,\nu)$ for given N, α and ν and calls `isconnected.mat` to determin the connectivity.

```
function[conn] = connected(N,alpha,nu)
R=2*log(N/nu);
cosh_R = cosh(R);
```

```
V = rand(N, 2);
RAM=550000000;

r=acosh(V(:,1)*(cosh(alpha*R)-1)+1)/alpha;
theta=V(:,2)*2*pi;

cosh_r = cosh(r);
cos_theta = cos(theta);
sin_theta = sin(theta);
r_S=sqrt((cosh_r-1)./(cosh_r+1));
x_S=r_S.*cos_theta;
y_S=r_S.*sin_theta;
norm=x_S.^2+y_S.^2;

S_help1 = [2*x_S.^2, 2*x_S, 2*ones(N,1),
    2*y_S.^2, 2*y_S, 2*ones(N,1)];
S_help1 = bsxfun(@rdivide, S_help1, 1-norm);
S_help1 = [ones(N,1), S_help1];
S_help2 = [ones(N,1),-2*x_S,x_S.^2,ones(N,1),-2*y_S, y_S.^2];
S_help2 = bsxfun(@rdivide, S_help2, 1-norm);
S_help2 = [ones(N,1), S_help2];

connMatrix = sparse(1:N,1:N,false);
inc=ceil(RAM/N);
i=0;
count=1;
fprintf('<');
while i+inc<N
```

```
    temp=sparse(S_help1(i+1:i+inc,:)*S_help2(i+1:end,:)'
                <cosh_R);
    connMatrix(i+1:i+inc,i+1:end) = temp;
    connMatrix(i+inc+1:end,i+1:i+inc) = temp(:,inc+1:end)';
    i=i+inc;
    inc=ceil(RAM/(N-i));
end
if(count*N<i*11)
    fprintf('.');
    count=count+1;
end
fprintf('>');
connMatrix(i+1:end,i+1:end)=
    S_help1(i+1:end,:)*S_help2(i+1:end,:)'<cosh_R;

conn=isconnected(connMatrix);
end
```

The function `findF.m` iterates `connected.m` a given number of times for a given N, α and ν.

```
function [count] =findF(N,alpha,nu,M)
count=0;
for i=1:M
    count=count+connected(N,alpha,nu);
end
count=count/M;
end
```

The function `approxF.m` iterates `findF.m` for a given range of α and assembles the results into a plot. The results are then saved.

```
function approxF(N, alpha, M, min, max, steps)
inc=(max-min)/(steps-1);
results=zeros(steps,2);
if steps == 1
    inc=0;
    min=(min+max)/2;
end
for i=1:steps
    temp=min+inc*(i-1);
    results(i,1)=temp;
    results(i,2)=findF(N,alpha,temp,M);
end
h = figure(1);
plot(results(:,1),results(:,2));
save(['results.mat'],'-v7.3');
end
```

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